The ultrafilter number and $\mathfrak{hm}$

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Abstract. The cardinal invariant $\mathfrak{hm}$ is defined as the minimum size of a family of $c_{\min}$-monochromatic sets that cover 2^ω (where $c_{\min}(x, y)$ is the parity of the biggest initial segment both x and y have in common). We prove that $\mathfrak{hm} = \omega_1$ holds in Shelah's model of $\mathfrak{i} < \mathfrak{u}$, so the inequality $\mathfrak{hm} < \mathfrak{u}$ is consistent with the axioms of ZFC. This answers a question of Thilo Weinert. We prove that the diamond principle $\Diamond_{\mathfrak{d}}$ also holds in that model.

1 Introduction

In [9], Geschke et al. defined the coloring $c_{\min} : [2^\omega]^2 \longrightarrow 2$ given by $c_{\min}(x, y) = 0$ if $\triangle(x, y)$ is even and $c_{\min}(x, y) = 1$ in case $\triangle(x, y)$ is odd (where $\triangle(x, y)$ is the length of the largest initial segment that x and y have in common). They defined the cardinal invariant $\mathfrak{hm}$ as the smallest size of a family of $c_{\min}$-monochromatic sets that covers 2^ω. Since every $c_{\min}$-monochromatic set is nowhere dense, it follows that $\text{cov}(\mathcal{M}) \leq \mathfrak{hm}$ (by $\text{cov}(\mathcal{M})$, we denote the smallest family of meager sets that are needed to cover 2^ω). However, the cardinal invariant $\mathfrak{hm}$ may be much larger than $\text{cov}(\mathcal{M})$:

Proposition 1 [7, 9] $\text{cof}(\mathcal{N}), \mathfrak{c}^- \leq \mathfrak{hm}$ *(where $\mathfrak{c}^- = \mathfrak{c}$ if $\mathfrak{c}$ is a limit cardinal and if $\mathfrak{c} = \kappa^+$ then $\mathfrak{c}^- = \kappa$).*

Above, $\text{cof}(\mathcal{N})$ denotes the cofinality of the ideal of Lebesgue-null subsets of 2^ω. Therefore, $\mathfrak{hm}$ is bigger than all the cardinal invariants that appear in the Cichoń diagram (see [2, 3]). On the other hand, it is known that the inequality $\mathfrak{hm} < \mathfrak{c}$ is consistent; in fact, it holds in the Sacks model [9] and in the Miller lite model [7]. It is interesting to compare it with the other cardinal invariants of the continuum. The following is an interesting open problem of Thilo Weinert:

Problem 2 (Weinert)   Is the inequality $\mathfrak{hm} < \mathfrak{r}$ consistent?

Where $\mathfrak{r}$ denotes the smallest size of a *reaping family*, i.e., the smallest size of a family $\mathcal{R} \subseteq [\omega]^\omega$ such that for every $A \in [\omega]^\omega$, there is $R \in \mathcal{R}$ such that either $R \subseteq^* A$ or $R \subseteq^* \omega \setminus A$. This problem is still open. Weinert made some interesting advances with respect to the question in his thesis (and it was after reading the first chapters of his thesis that the author became interested on the topic). In his thesis, he asked the following:

Problem 3 (Weinert)   Is the inequality $\mathfrak{hm} < \mathfrak{u}$ consistent?

Where 𝔲 denotes the *ultrafilter number*, the smallest size of a base of an ultrafilter[1] in ω. In this note, we will provide a positive answer to the question. Since $\mathfrak{r} \leq \mathfrak{u}$ our solution to may be viewed as a partial solution to the first problem.

In [13], Shelah built a model of $\mathfrak{i} < \mathfrak{u}$ and we will prove that ℏ𝔪 $< \mathfrak{u}$ holds in that model too. A variant of the forcing of Shelah was recently used in [5] by Chodounský et al. in order to show that the inequality $\mathfrak{f} < \mathfrak{u}$ is consistent ($\mathfrak{f}$ is the *free sequence number* introduced by Monk, the reader may consult the interesting paper [5] for the definition of $\mathfrak{f}$ and will not be used in here).

Recall that an infinite family $\mathcal{A} \subseteq [\omega]^\omega$ is an *almost disjoint family* (AD) if the intersection of any two of its elements is finite and $\mathcal{A}$ is MAD if it is maximal with this property. The *almost disjointness number* $\mathfrak{a}$ is defined as the smallest size of a MAD family. In the last part of the paper, we will prove that $\mathfrak{a} = \omega_1$ holds in the model of Shelah. In order to prove that there is small MAD family in such model, we prove that Hrušák diamonds $\Diamond_\mathfrak{d}$ holds in there. This is enough since Hrušák proved that $\Diamond_\mathfrak{d}$ implies that $\mathfrak{a} = \omega_1$.

We would like to point out that the following problem of Weinert is still open:

Problem 4 (Weinert)   Is the inequality ℏ𝔪 $< \mathfrak{a}$ consistent?

This problem seems very hard since the only known method to construct models of $\mathrm{cof}(\mathcal{N}) < \mathfrak{a}$ is with the aid of templates (see [4, 15]) but this method does not seem to help with the question.

In a recent work with Cruz-Chapital et al., we obtained a different prove that $\mathfrak{a} = \omega_1$ holds in the model of Shelah mentioned above. In [12], it is proved that the Shelah forcing strongly preserves the tightness of MAD families, which is a property that guarantees us that certain type of MAD families are preserved (even in the iteration). The proof from [12] and the one from this paper are complementary. On the one hand, the argument of [12] is much simpler and it proves that there are tight MAD families on the Shelah model. On the other hand, we get that $\Diamond_\mathfrak{d}$ holds in that model.

We would like to mention that although the cardinal invariant ℏ𝔪 is not very well known, it is very interesting. We would like to mention some of the motivations for studying this invariant. Given a polish space X and a continuous coloring $c : [X]^2 \longrightarrow 2$, define ℏ𝔪$(c)$ as the smallest size of a family of c-monochromatic sets that cover X. Note that ℏ𝔪 $=$ ℏ𝔪$(c_{\min})$. It turns out that ℏ𝔪 and $c_{\min}$ are critical in the study of continuous colorings, if $c : [X]^2 \longrightarrow 2$ is a continuous coloring such that ℏ𝔪(c) is uncountable, then ℏ𝔪 $\leq$ ℏ𝔪(c)[2]. Furthermore, the study of the cardinal invariants for continuous colorings has important implications in the study of the structure of nonconvex closed subsets of the Euclidean plane. This and more very interesting and deep results can be consulted in [7–9].

Our notation is mostly standard, with one important exemption. A tree $T \subseteq X^{<\omega}$ is a set closed under taking subsequences and the *set of branches of T* (denoted by $[T]$) is the set $\{f \in X^\omega \mid \forall n \in \omega \, (f \restriction n \in T)\}$. If $s, t \in X^{<\omega}$, by $s \hat{\ } t$, we denote the concatenation of s and t, if $x \in X$, we will often write $s \hat{\ } x$ instead of $s \hat{\ } \langle x \rangle$. Given a tree $T \subseteq X^{<\omega}$ and $s \in T$, define $suc_T(s) = \{x \in X \mid s \hat{\ } x \in T\}$. Given $n \in \omega$, the *n-level*

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[1]For us, all ultrafilters are nonprincipal.

[2]There is also a continuous coloring $c_{\max} : [2^\omega]^2 \longrightarrow 2$ such that ℏ𝔪$(c) \leq$ ℏ𝔪$(c_{\max})$.

of T is defined as $T_n = \{ s \in T \mid |s| = n+1 \}$[3] and let $T_{\leq n} = \bigcup_{i \leq n} T_i$ as well as $T_{<n} = \bigcup_{i < n} T_i$.
Note that if $s = \langle x_0, \ldots, x_n \rangle \in T$, then $s \in T_n$.

2 Shelah's forcing with respect to an ultrafilter

Given a partial order $\mathbb{P}$ and an ultrafilter $\mathcal{U}$, we say that $\mathbb{P}$ *ultradestroys* $\mathcal{U}$ if $\mathbb{P}$ forces that $\mathcal{U}$ is no longer the base for an ultrafilter. In [13], Shelah designed a forcing to ultradestroy any given ultrafilter while causing "minimal damage" to the ground model.

Definition 5   Let $\mathcal{U}$ be an ultrafilter. A set $E = \{ E_n \mid n \in \omega \} \subseteq \wp(\omega)$ is called an $\mathcal{U}$-*partition* if E satisfies the following properties:

1. E is a pairwise disjoint family of elements of $\mathcal{U}^*$ (where $\mathcal{U}^*$ denotes the dual ideal of $\mathcal{U}$, i.e., $\mathcal{U}^* = \{ \omega \setminus A \mid A \in \mathcal{U} \}$).
2. $dom(E) = \bigcup_{n \in \omega} E_n \in \mathcal{U}$.
3. Each E_n is nonempty.

In other words, E is an $\mathcal{U}$-partition if E is a partition of some element in $\mathcal{U}$ such that all of its classes are in $\mathcal{U}^*$. Given $a \in dom(E)$ we define $[a]_E$ as the unique $E_n \in E$ such that $a \in E_n$. Define $A_E = \{ a_E(n) \mid n \in \omega \}$, where $a_E(n) = min(E_n)$ and this set will be called the *leaders of* E. We will always assume that if $E = \{ E_n \mid n \in \omega \}$ is an $\mathcal{U}$-partition and $n < m$, then $a_E(n) < a_E(m)$.

Definition 6   Let E and E' be two $\mathcal{U}$-partitions, we say that $E' <_{part} E$ if every E'-class is the union of E-classes.

Equivalently, E' can be constructed by throwing away entire classes of E and merging (i.e., taking unions) some classes of E (obviously, making sure that the new classes are elements of $\mathcal{U}^*$). Note that if $E' <_{part} E$ then $dom(E') \subseteq dom(E)$ and $A_{E'} \subseteq A_E$.

Definition 7   Let $\mathcal{U}$ be an ultrafilter. The *Shelah forcing with respect to* $\mathcal{U}$ (denoted by $\mathbb{S}(\mathcal{U})$) is the set of all $p = \left(E_p, H_p, Lov_p, Hat_p \right)$ with the following properties:

1. E_p is a $\mathcal{U}$-partition (let $A_p = A_{E_p}$, $a_p(n) = a_{E_p}(n)$ and $[a_p(n)]_p = [a_p(n)]_{E_p}$).
2. If $n \in \omega$ then $\{ Lov_p(a_p(n)), Hat_p(a_p(n)) \}$ is a partition of $[a_p(n)]_p$ and $a_p(n) \in Lov_p(a_p(n))$.
3. $Lov_p = \{ Lov_p(a_p(n)) \mid n \in \omega \}$ and $Hat_p = \{ Hat_p(a_p(n)) \mid n \in \omega \}$.
4. If $m \notin dom(E_p)$ then $H_p^m : 2^{A_p \cap m} \longrightarrow 2$.
5. $H_p = \{ H_p^m \mid m \notin dom(E_p) \}$.

    Given $p, q \in \mathbb{S}(\mathcal{U})$, define $p \leq q$ if the following holds:

1. $E_p <_{part} E_q$ (so $A_p \subseteq A_q$).
2. If $a \in A_p$ (so $a \in A_q$ and its class was not thrown away) then $Lov_q(a) \subseteq Lov_p(a)$ and $Hat_q(a) \subseteq Hat_p(a)$.

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[3] T_n is often defined as $\{ s \in T \mid |s| = n \}$, but for our purposes, it will be better to define it the way we did.

3. If $a \in A_q$ and there is $b \in A_p$ such that $b < a$ and $a \in [b]_p$ (so the class of a was merged with a previous class) then we have the following:
   (a) If $a \in Lov_p(b)$ then $Lov_q(a) \subseteq Lov_p(b)$ and $Hat_q(a) \subseteq Hat_p(b)$.
   (b) If $a \in Hat_p(b)$ then $Lov_q(a) \subseteq Hat_p(b)$ and $Hat_q(a) \subseteq Lov_p(b)$.
4. If $a \in A_q \backslash dom(E_p)$ and $n \in [a]_q$ then $H_p^n : 2^{A_p \cap n} \longrightarrow 2$ is defined as follows:
   (a) H_p^a may be any function (with domain $2^{A_p \cap n}$ and codomain 2 of course).
   (b) If $f \in 2^{A_p \cap n}$ and $n \in Lov_q(a)$ then $H_p^n(f) = H_p^a(f \upharpoonright a)$ and if $n \in Hat_q(a)$ then $H_p^n(f) = 1 - H_p^a(f \upharpoonright a)$.
   Let $n \notin dom(E_q)$ and $f : A_p \cap n \longrightarrow 2$. Define $\overline{f} : A_q \cap n \longrightarrow 2$ as follows:

$$
\overline{f}(b) = \begin{cases}
f(b) & \text{if} \quad b \in A_p \\
f(a) & \text{if} \quad b \notin A_p \text{ and there is } a \in A_p \cap n \\
& \quad \text{with } b \in Lov_p(a) \\
1 - f(a) & \text{if} \quad b \notin A_p \text{ and there is } a \in A_p \cap n \\
& \quad \text{with } b \in Hat_p(a) \\
H_p^b(f \upharpoonright (A_p \cap b)) & \text{if} \quad b \notin dom(E_p) \\
& \quad \text{(as defined in point 4 above)}
\end{cases}
$$

5. $H_p^n(f) = H_q^n(\overline{f})$ for every $n \notin dom(E_q)$.

Point 5 in the definition of $p \leq q$ might be the hardest to understand. We can see an example. Assume that $0, 1, 2, 3 \in A_q$ and $4 \notin dom(E_q)$. Let $p \leq q$ such that $0, 2 \in A_p$ and $1 \in Hat_p(0)$ while $3 \in Lov_p(2)$. Let $g, h \in 2^{A_p \cap 4}$ given by $g(0) = 0$ and $g(2) = 1$, while $h(0) = 1$ and $h(2) = 1$. In this case, we have that $\overline{g} = \{(0,0), (1,1), (2,1), (3,1)\}$ and $\overline{h} = \{(0,1), (1,0), (2,1), (3,1)\}$.

In general, we need to argue that H_p^n is well-defined (i.e., if $f : A_p \cap n \longrightarrow 2$, then $\overline{f} : A_q \cap n \longrightarrow 2$, so we are allowed to apply H_q^n to $\overline{f}$). We prove this in the following lemma:

Lemma 8 *Let $\mathcal{U}$ be an ultrafilter and $p, q \in \mathbb{S}(\mathcal{U})$ that satisfy points 1–4 of the definition of $p \leq q$. If n is a natural number such that $n \notin dom(E_q)$, then H_p^n is well-defined.*

Proof Let $n \in \omega$ such that $n \in \omega \backslash dom(E_q)$ and $f : A_p \cap n \longrightarrow 2$, we need to prove that $\overline{f}$ is well-defined. Let $b \in A_q \cap n$, we need to argue that $\overline{f}(b)$ is actually defined. We proceed by cases:

Case 9 $b \in dom(E_p)$.

Here, $\overline{f}(b)$ is defined by one of the first three clauses in the definition of $\overline{f}$ (if b is a leader in E_p, then the first clause applies, otherwise, either clause 2 or 3 applies).

Case 10 $b \notin dom(E_p)$.

Here, we have that $\overline{f}(b) = H_p^b(f \upharpoonright (A_p \cap b))$ and this is well-defined by point 4 of the definition of $p \leq q$. ∎

Now, we define the following:

Definition 11   Let $G \subseteq \mathbb{S}(\mathcal{U})$ be a generic filter. In $V[G]$, we define *the generic real* x_{gen} as the set of all $n \in \omega$ such that there is $p \in G$ for which $n \notin dom(E_p)$ and H_p^n is the constant 1 function.

We would like to mention that the term *generic real* may not be completely justified. We do not know if the generic filter G can be reconstructed from x_{gen}.[4]

Recall that if $p \in \mathbb{S}(\mathcal{U})$ and $n \notin dom(E_p)$ then the domain of H_p^n is $2^{A_p \cap n}$. Since we can always throw away a given class, for every $p \in \mathbb{S}(\mathcal{U})$ and for every $n \in \omega$, there is $q \leq p$ such that $n \notin dom(E_q)$ and H_q^n is a constant function (this is the case if $A_q \cap n$ is empty). The following lemma follows from the definitions:

Lemma 12 [13]   Let $p \in \mathbb{S}(\mathcal{U})$, $a \in A_p$ and $n \notin dom(E_p)$.

1. $p \Vdash$ " $\left(Lov_p(a) \subseteq \dot{x}_{gen} \right) \vee \left(Lov_p(a) \cap \dot{x}_{gen} = \emptyset \right)$."
2. $p \Vdash$ " $\left(Hat_p(a) \subseteq \dot{x}_{gen} \right) \vee \left(Hat_p(a) \cap \dot{x}_{gen} = \emptyset \right)$."
3. There are $q_0, q_1 \leq p$ such that the following holds: $q_0 \Vdash$ " $Lov_p(a) \subseteq \dot{x}_{gen}$ " and $q_1 \Vdash$ " $Lov_p(a) \cap \dot{x}_{gen} = \emptyset$."
4. There are $q_0, q_1 \leq p$ such that the following holds: $q_0 \Vdash$ " $Hat_p(a) \subseteq \dot{x}_{gen}$ " and $q_1 \Vdash$ " $Hat_p(a) \cap \dot{x}_{gen} = \emptyset$."
5. H_p^n is the constant 1 function if and only if $p \Vdash$ " $n \in \dot{x}_{gen}$."
6. H_p^n is the constant 0 function if and only if $p \Vdash$ " $n \notin \dot{x}_{gen}$."

Now it is possible to conclude the following:

Lemma 13 [13] $\mathbb{S}(\mathcal{U})$ ultradestroys $\mathcal{U}$, in fact, both $\dot{x}_{gen}$ and $\omega \backslash \dot{x}_{gen}$ are forced to have infinite intersection with every element of $\mathcal{U}$.

Proof   We will prove that $\dot{x}_{gen}$ is forced to intersect every element of $\mathcal{U}$, the proof for $\omega \backslash \dot{x}_{gen}$ is similar. Let $B \in \mathcal{U}$, $p \in \mathbb{S}(\mathcal{U})$ and $n \in \omega$. It is enough to prove that there is $q \leq p$ such that $q \Vdash$ " $B \cap \dot{x}_{gen} \not\subseteq n$." Since $dom(E_p) \in \mathcal{U}$, there is $m > n$ such that $m \in B \cap dom(E_p)$. Since $m \in dom(E_p)$, we know there is a (unique) $a \in A_p$ such that $m \in [a]_p$. Define a condition q with the following properties:

1. $q \leq p$.
2. $dom(E_q) = dom(E_p) \backslash [a]_p$.
3. If $b \in dom(E_p) \backslash [a]_p$ then $[b]_p = [b]_q$.
4. If $m \in Lov_p(a)$ then H_q^a is the constant 1 function, and if $m \in Hat_p(a)$, then H_q^a is the constant 0 function.

It is easy to see that $q \Vdash$ " $m \in B \cap \dot{x}_{gen}$," so we are done.   ∎

The following definitions will be frequently used:

Definition 14   Let $p, q \in \mathbb{S}(\mathcal{U})$ and $n \in \omega$. We define the following orders on $\mathbb{S}(\mathcal{U})$:

1. $p \leq_n q$ if $p \leq q$ and $\left[a_p(i) \right]_p = \left[a_q(i) \right]_q$ for every $i \leq n$.
2. $p \leq_n^* q$ if $p \leq_{n-1} q$ and $a_p(n) = a_q(n)$.[5]
3. $p \leq_n^{**} q$ if $p \leq_n^* q$ and $dom(E_p) = dom(E_q)$.

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[4] This remark was pointed to us by one of the referees.
[5] By $p \leq_{-1} q$, we simply mean $p \leq q$.

In other words, $p \leq_n q$, if p extends q and the first n-classes of E_q do not get thrown away nor they merge with other classes. Meanwhile, $p \leq_n^* q$ if $p \leq_{n-1} q$ and the n-class of E_q was not thrown away (but it could have been merge with other later classes). In a similar fashion, $p \leq_n^{**} q$ means that $p \leq_n^* q$ and no class was thrown away. Note that $p \leq_0^* q$ means that p extends q and the 0-class was not thrown away. It is easy (but not automatic) to see that these relations are transitive and symmetric. Note that if $n < m$ and $p \leq_m q$, then $p \leq_n q$. A similar remark apply for the other relations.

Definition 15 Let $q, r, q', r' \in \mathbb{S}(\mathcal{U})$ and $n \in \omega$. We say $\langle q, r, q', r' \rangle$ is an *n-nice sequence* if the following holds:

1. $q \perp r$ (i.e., q and r are incompatible).
2. If $n < i$ then $\left[a_q(i)\right]_q = \left[a_r(i)\right]_r$.
3. $q' \leq_n^* q$ and $r' \leq_n^* r$.
4. If $n + 1 < i$ then $\left[a_{q'}(i)\right]_{q'} = \left[a_{r'}(i)\right]_{r'}$.
5. $dom\left(E_q\right) \setminus dom\left(E_{q'}\right) \subseteq \left[a_{r'}(n)\right]_{r'}$ and $dom\left(E_r\right) \setminus dom\left(E_{r'}\right) \subseteq \left[a_{q'}(n)\right]_{q'}$.

The incompatibility of q and r is not required for most arguments. However, most of time we want to work using nice sequences, they will already be incompatible from the beginning. Note that $\langle q, r, q, r \rangle$ is an n-nice sequence if and only if q and r are incompatible and if $n < i$ then $\left[a_q(i)\right]_q = \left[a_r(i)\right]_r$.

We will also need the following notions:

Definition 16 Let $p \in \mathbb{S}(\mathcal{U}), n \in \omega$ and $D \subseteq \mathbb{S}(\mathcal{U})$.

1. D is *$\leq_n$-dense below p* if for every $q \leq_n p$ there is $r \leq_n q$ such that $r \in D$.
2. D is *$\leq_n$-open below p* if for every r such that $r \leq_n p$ and $r \in D$ then $q \in D$ whenever $q \leq_n r$.
3. D is *$\leq_n$-open dense below p* if it is both $\leq_n$-open and $\leq_n$-dense below p.
4. The same definitions apply for $\leq_n^*$.

In the above definitions, if D is $\leq_n$-dense below $1_{\mathbb{S}(\mathcal{U})}$ (where $1_{\mathbb{S}(\mathcal{U})}$ is the largest element of $\mathbb{S}(\mathcal{U})$), then we will simply write "D is $\leq_n$-dense." Similar conventions are adopted for the other notions. Expressions like "D is $\leq_n^*$-open and $\leq_m^*$-dense below p," actually mean "D is $\leq_n^*$-open below p and $\leq_m^*$-dense below p."

Note that if $n < m$ and D is $\leq_n$-open below p, then D is $\leq_m$-open below p. The following lemma is the base for several constructions in the paper:

Lemma 17 *Let $n \in \omega$ and q, r be two conditions such that $\langle q, r, q, r \rangle$ is an n-nice sequence. Let D_q be an $\leq_n^*$-open dense set below q and D_r be an $\leq_n^*$-open dense set below r. There are $q' \in D_q$ and $r' \in D_r$ such that $\langle q, r, q', r' \rangle$ is an n-nice sequence.*

Proof We first find $q_1 \leq_n^* q$ such that $q_1 \in D_q$. Let $S = \left(dom\left(E_q\right) \setminus dom\left(E_{q_1}\right)\right) \cup \left(\left[a_{q_1}(n)\right]_{q_1} \setminus \left[a_q(n)\right]_q\right)$ and note it is an element of $\mathcal{U}^*$. Let r_1 be any condition such that:

1. $r_1 \leq_n^* r$.
2. $\left[a_{r_1}(n)\right]_{r_1} = \left[a_r(n)\right]_r \cup S$.
3. $dom\left(E_{r_1}\right) = dom\left(E_r\right)$.

4. E_{r_1} and E_{q_1} are equal on $\bigcup\{[a_{q_1}(i)]_{q_1} \mid n < i\}$ (i.e., for every $x \in \bigcup\{[a_{q_1}(i)]_{q_1} \mid n < i\}$, we have that $[x]_{E_{r_1}} = [x]_{E_{r_2}}$, recall that if $n < i$ then $[a_q(i)]_q = [a_r(i)]_r$).

Now we find $r_2 \leq_n^* r_1$ (so $r_2 \leq_n^* r$) such that $r_2 \in D_r$. Let $Z = (dom(r_1)\setminus dom(r_2)) \cup ([a_{r_2}(n)]_{r_2} \setminus [a_{r_1}(n)]_{r_1})$ which is also an element of $\mathcal{U}^*$. Let q_2 be any condition such that:

1. $q_2 \leq_n^* q_1$.
2. $[a_{q_2}(n)]_{q_2} = [a_{q_1}(n)]_{q_1} \cup Z$.
3. $dom(E_{q_2}) = dom(E_{q_1})$.
4. E_{q_2} and E_{r_2} are equal on $\bigcup\{[a_{r_2}(i)]_{r_2} \mid n+1 < i\}$ (in the same sense as above).

Since $q_2 \leq_n^* q_1$ and D_q is $\leq_n^*$-open, it follows that $q_2 \in D_q$. It is clear that $\langle q, r, q_2, r_2 \rangle$ has the desired properties.   ■

The axiom A structure in a forcing is very useful. Unfortunately, it does not seem that $\mathbb{S}(\mathcal{U})$ has one (however, in [13] it was proved that $\mathbb{S}(\mathcal{U})$ is $< \omega_1$-proper, so by a theorem of Tetsuya Ishiu, [11] $\mathbb{S}(\mathcal{U})$ is forcing equivalent to a partial order with an axiom A structure. Unfortunately, we were unable to take advantage of this result). The purpose of the following definitions and results are to obtain a similar structure to the one of an axiom A forcing.

Definition 18   We say $T = \langle q_i, r_i \rangle_{i<\omega}$ is a 1-*fusion sequence* if the following holds:

1. q_0 and r_0 are incompatible but $E_{q_0} = E_{r_0}$.
2. If $i < \omega$ then $\langle q_i, r_i, q_{i+1}, r_{i+1} \rangle$ is an i-nice sequence.

We will say $T = \langle q_i, r_i \rangle_{i \leq n+1}$ is a 1-*finite fusion sequence* if it satisfy the previous points for every $i < n + 1$.

Definition 19   Let $\langle p_i \rangle_{i \in \omega} \subseteq \mathbb{S}(\mathcal{U})$ such that $p_{i+1} \leq_i^* p_i$ for every $i \in \omega$. We define *the limit of* $\langle p_i \rangle_{i \in \omega}$ as $Lim(\langle p_i \rangle_{i \in \omega}) = p = (E_p, H_p, Lov_p, Hat_p)$ as follows:

1. $dom(E_p) = \bigcup_{i<\omega} [a_{p_{i+1}}(i)]_{p_{i+1}}$.
2. $[a_p(i)]_{E_p} = [a_{p_{i+1}}(i)]_{p_{i+1}}$.
3. $Lov_p(a_p(i)) = Lov_{p_{i+1}}(a_{p_{i+1}}(i))$.
4. $Hat_p(a_p(i)) = Hat_{p_{i+1}}(a_{p_{i+1}}(i))$.
5. If $m \notin dom(E_p)$ then $H_p^m = H_{p_m}^m$.

Above we are not claiming that the limit is a condition. In fact, $p = Lim(\langle p_i \rangle_{i \in \omega})$ may not be in $\mathbb{S}(\mathcal{U})$ since it might be the case that $dom(E_p)$ is not an element of $\mathcal{U}$. Nevertheless, if p is indeed a condition, then $p \leq_i^* p_i$ for every $i \in \omega$. The following result plays the role of a fusion sequence in an Axiom A forcing:

Lemma 20   *If $T = \langle q_i, r_i \rangle_{i<\omega}$ is a 1-fusion sequence in $\mathbb{S}(\mathcal{U})$ then there is $\overline{p}$ that either is a lower bound of $\langle q_i \rangle_{i<\omega}$ or it is a lower bound of $\langle r_i \rangle_{i<\omega}$. In fact, either $Lim(\langle q_i \rangle_{i \in \omega}) \in \mathbb{S}(\mathcal{U})$ or $Lim(\langle r_i \rangle_{i \in \omega}) \in \mathbb{S}(\mathcal{U})$.*

Proof   Assume $q = Lim(\langle q_i \rangle_{i \in \omega}) \notin \mathbb{S}(\mathcal{U})$, we will that $r = Lim(\langle r_i \rangle_{i \in \omega})$ is a condition in $\mathbb{S}(\mathcal{U})$. In order to show this, we must first argue that $dom(E_{q_0}) = dom(E_q) \cup dom(E_r)$. If $n \in dom(E_{q_0})\setminus dom(E_q)$, we then may

find $i \in \omega$ such that $n \in dom(q_i) \setminus dom(q_{i+1})$. Since $\langle q_i, q_{i+1}, r_i, r_{i+1} \rangle$ is i-nice then $n \in [a_{r_{i+1}}(i)]_{r_{i+1}} \cup [a_{r_{i+1}}(i+1)]_{r_{i+1}}$ so $n \in dom(E_r)$. Since $dom(E_{q_0}) \in \mathcal{U}$ and $\mathcal{U}$ is an ultrafilter, it must be the case that $dom(E_r) \in \mathcal{U}$. ∎

In the above case, we would say that $Lim(\langle q_i \rangle_{i \in \omega})$ (or $Lim(\langle r_i \rangle_{i \in \omega})$ in case $Lim(\langle q_i \rangle_{i \in \omega}) \notin \mathbb{S}(\mathcal{U})$) *is a fusion for T*. For the case of $\leq^{**}$ the situation is simpler:

Lemma 21   *If* $\langle p_i \rangle_{i \in \omega} \subseteq \mathbb{S}(\mathcal{U})$ *is a sequence such that* $p_{i+1} \leq_i^{**} p_i$ *for every* $i \in \omega$ *then there is* $q \in \mathbb{S}(\mathcal{U})$ *such that* $q \leq_i^{**} p_i$ *for every* $i \in \omega$.

Proof   We claim that $q = Lim(\langle p_i \rangle_{i \in \omega})$ is as desired. We only need to prove that q is really a condition. In order to achieve that, it is enough to prove that $dom(E_q) \in \mathcal{U}$. Since the sequence $\langle p_i \rangle_{i \in \omega}$ is $\leq_i^{**}$-decreasing, it follows that $dom(E_q) = dom(E_{p_0})$ and we are done. ∎

We will need the following definition:

Definition 22   Let $p \in \mathbb{S}(\mathcal{U})$, $n > 0$ and $h : n \longrightarrow 2$. We define $p[h]$ as the condition extending p with the following properties:

1. $Dom(E_{p[h]}) = Dom(E_p) \setminus \bigcup_{i < n} [a_p(i)]_{E_p}$.
2. If $m \in Dom(E_{p[h]})$ then $[m]_{p[h]} = [m]_p$.
3. If $a \in A_{p[h]}$ then $Lov_{p[h]}(a) = Lov_p(a)$ and $Hat_{p[h]}(a) = Hat_p(a)$.
4. If $i < n$ then $H_{p[h]}^{a_p(i)}$ is the constant function with value $h(i)$.

In other words, $p[h]$ is obtained by throwing out the first n-classes and tell their leaders to "follow h."

The following lemma (which we leave to the reader) is very easy, yet it will be often used in future arguments:

Lemma 23   *Let* $n \in \omega$ *and* $p, q \in \mathbb{S}(\mathcal{U})$ *such that* $q \leq_n^* p$. *If* $h \in 2^{n+1}$, *then* $q[h] \leq p[h]$.

We have the following:

Lemma 24   *Let* $p \in \mathbb{S}(\mathcal{U})$ *and* $n > 0$. *The set* $\{p[h] \mid h \in 2^n\}$ *is a maximal antichain below* p.

Proof   Let $h, g \in 2^n$ with $h \neq g$, we will see that $p[h]$ and $p[g]$ are incompatible. Let $i < n$ such that $h(i) = 0$ and $g(i) = 1$. In this way, we have that $p[h] \Vdash ``i \notin \dot{x}_{gen}"$ and $p[g] \Vdash ``i \in \dot{x}_{gen},"$ so this two conditions are incompatible.

We now need to prove that the set $\{p[h] \mid h \in 2^n\}$ is predense below p. Let $q \leq p$, take $r \leq q$ such that for every $i < n$, either $r \Vdash ``i \in \dot{x}_{gen}"$ or $r \Vdash ``i \notin \dot{x}_{gen}."$ Define $h : n \longrightarrow 2$ such that $h(i) = 1$ if and only if $r \Vdash ``i \in \dot{x}_{gen}."$ It follows that $r \leq p[h]$. ∎

We now define the following,

Definition 25   Let $D \subseteq \mathbb{S}(\mathcal{U})$ be an open dense set below p and $n > 0$. We define $\widetilde{D}(p, n) = \{q \leq_n^* p \mid \forall h \in 2^n (q[h] \in D)\}$.

We now have the next result,

Lemma 26   *If* $D \subseteq \mathbb{S}(\mathcal{U})$ *is an open dense set below* p *and* $n > 0$, *then* $\widetilde{D}(p, n)$ *is* $\leq_n^*$-*open dense below* p.

Proof   We first show that $\widetilde{D}(p, n)$ is $\leq_n^*$-dense below p. Let $q \leq_n^* p$ and enumerate $2^n = \{h_i \mid i < k\}$, we can then recursively find a sequence $\langle q_i \rangle_{i < k+1}$ with the following properties:

1. $q_0 = q$.
2. $\langle q_i \rangle_{i < k+1}$ is $\leq_n^*$-decreasing.
3. $q_{i+1}[h_i] \in D$.

It is then easy to see that $q_{k+1} \in \widetilde{D}(p, n)$. Finally, $\widetilde{D}(p, n)$ is $\leq_n^*$-open, since whenever $r \leq_n^* q$ then $r[h] \leq q[h]$ for every $h \in 2^n$.   ∎

With these results, we can finally prove that Shelah's forcing is proper.

Proposition 27 [13]   *If $\mathcal{U}$ is an ultrafilter, then $\mathbb{S}(\mathcal{U})$ is a proper forcing.*

Proof   We will prove that $\mathbb{S}(\mathcal{U})$ is proper. Let M be a countable elementary submodel of some $H(\kappa)$ and $p \in M$. Let $\{D_n \mid n \in \omega\}$ enumerate all open dense subsets of $\mathbb{S}(\mathcal{U})$ that belong to M. For every $n \in \omega$ let $\widetilde{D}_n(p, n) = \{q \leq_n^* p \mid \forall h \in 2^n \, (q[h] \in D_n)\}$. We know that $\widetilde{D}_n(p, n)$ is $\leq_n^*$-open dense. It is also clear that each $\widehat{D}_n(p, n)$ is an element of M. Using Lemma 17, we can construct $T = \langle q_i, r_i \rangle_{i < \omega}$ a 1-fusion sequence with the following properties:

1. $T \subseteq M$.
2. $q_0, r_0 \leq p$.
3. $q_{i+1} \in \widetilde{D}_i(q_i, i)$ and $r_{i+1} \in \widetilde{D}_i(r_i, i)$ for every $i \in \omega$.

It is easy to see that if $\overline{p}$ is a fusion of T, then $\overline{p}$ will be an $(M, \mathbb{S}(\mathcal{U}))$-generic condition extending p.   ∎

Recall that a forcing $\mathbb{P}$ has the *Sacks property* if for every $p \in \mathbb{P}$ and $\dot{f}$ such that $p \Vdash ``\dot{f} \in \omega^\omega$," there are $q \leq p$ and a sequence $S = \langle S_n \rangle_{n \in \omega} \in V$ such that $S_n \in [\omega]^{2^{n+1}}$ and $q \Vdash ``\dot{f}(n) \in S_n$" for every $n \in \omega$. We will soon prove that $\mathbb{S}(\mathcal{U})$ has the Sacks property.

Definition 28   Let $\mathbb{P}$ be a partial order, $p \in \mathbb{P}$ and $\dot{x}$ a $\mathbb{P}$-name for an element of ω^ω. Define $\dot{x}[p] = \bigcup \{t \in \omega^{<\omega} \mid p \Vdash ``t \subseteq \dot{x}"\}$.

In case $\dot{x}$ is forced to be a new element of ω^ω (which is often the case we are interested in), we will have that $\dot{x}[p] \in \omega^{<\omega}$.

Definition 29   Let $p \in \mathbb{S}(\mathcal{U})$ and $\dot{x}$ such that $p \Vdash ``\dot{x} \in \omega^\omega$." We say p is $\dot{x}$-*nice* if $p[h]$ determines $\dot{x} \restriction n$ for every $h : n \longrightarrow 2$.

We now have the following:

Lemma 30 [13]   *Let $\dot{f}$ be a $\mathbb{S}(\mathcal{U})$-name for a new element of ω^ω. The set of all $\dot{f}$-nice conditions is a dense set.*

Proof   Let $p \in \mathbb{S}(\mathcal{U})$ and $n \in \omega$. Define $D_n = \{q \leq p \mid \exists s \in \omega^{n+1} (q \Vdash ``\dot{f} \restriction (n+1) = s")\}$. Clearly D_n is an open dense set below p, so by Lemma 26, we get that $\widetilde{D}_n(n, p)$ is $\leq_n^*$-open dense.

Let $q_0, r_0 \leq p$ be two incompatible conditions with $E_{q_0} = E_{r_0}$. Using the previous remark and Lemma 17, we can construct a 1-fusion sequence $T = \langle q_i, r_i \rangle_{i \in \omega}$ such that $q_{i+1}, r_{i+1} \in \widetilde{D}_i(i, p)$ for every $i \in \omega$. By Lemma 20, let $\overline{p}$ be a fusion of T. It follows that $\overline{p} \leq p$ and $\overline{p}$ is $\dot{f}$-nice.   ∎

With this result, we can conclude the following:

Proposition 31 [13]   *If* $\mathcal{U}$ *is an ultrafilter, then* $\mathbb{S}(\mathcal{U})$ *has the Sacks property.*

Proof   Let $\dot{f}$ be an $\mathbb{S}(\mathcal{U})$-name for a new real and $p \in \mathbb{S}(\mathcal{U})$. By the previous result, we can find $q \le p$ that is $\dot{f}$-nice. For every $n \in \omega$, define $S_n = \{m \mid \exists h \in 2^{n+1}(q[h] \Vdash$ "$\dot{f}(n) = m$")$\}$. It follows that S_n has size at most 2^{n+1} and $q \Vdash$ "$\dot{f}(n) \in S_n$." This finishes the proof. ∎

Most of the arguments in this section were based on the ones of [13]. Instead of our fusion sequences, Shelah used games to prove the properness and the Sacks property. We decided to use the fusion sequences instead of games because it will be easier to work with sequences rather than with games when dealing with the iteration.

3   Forcing with $\mathbb{S}(\mathcal{U})$ preserves $c_{\min}$-covering

It is easy to see that if $X \subseteq 2^\omega$ is $c_{\min}$-monochromatic, then the closure $\overline{X}$ is also $c_{\min}$-monochromatic. In this way, $\mathfrak{hm}$ is the smallest size of a family of $c_{\min}$-monochromatic closed sets that covers 2^ω. We will say that a tree $T \subseteq 2^{<\omega}$ is $c_{\min}$-*monochromatic* if $[T]$ is $c_{\min}$-monochromatic. This means that T either only splits at odd levels or at even levels.

Definition 32   We say that a forcing notion $\mathbb{P}$ *preserves* $c_{\min}$-*covering* if for every $p \in \mathbb{P}$ and for every $\mathbb{P}$-name $\dot{x}$ for a real in 2^ω, there are $q \le p$ and $T \in V$ a $c_{\min}$-monochromatic tree such that $q \Vdash$ "$\dot{x} \in [T]$."

Recall that if $\mathbb{P}$ is a partial order, $p \in \mathbb{P}$ and $\dot{x}$ a $\mathbb{P}$-name for a new element of 2^ω, we defined $\dot{x}[p] = \bigcup \{t \in 2^{<\omega} \mid p \Vdash$ "$t \subseteq \dot{x}$"$\}$. Since $\dot{x}$ is forced to be a new real, then $\dot{x}[p] \in 2^{<\omega}$. Note that for every $p \in \mathbb{P}$ there are $q, r \le p$ such that $\dot{x}[q]$ and $\dot{x}[r]$ are incompatible elements of $2^{<\omega}$. In this section, we will prove that Shelah's forcing $\mathbb{S}(\mathcal{U})$ preserves $c_{\min}$-covering. We will deal with the iteration in later sections. For the rest of this section, fix $\dot{x}$ a $\mathbb{S}(\mathcal{U})$-name for a new real. Recall that a condition p is $\dot{x}$-nice if $p[h]$ determines $\dot{x} \restriction n$ for every $h : n \longrightarrow 2$.

Definition 33   Let $p \in \mathbb{S}(\mathcal{U})$, define $S_p(\dot{x}) = \bigcup \{\dot{x}[p[h]] \mid h \in 2^{<\omega}\}$.

It is easy to see that if p is $\dot{x}$-nice, then $S_p(\dot{x})$ is a tree and $p \Vdash$ "$\dot{x} \in [S_p(\dot{x})]$." Recall that a tree $p \subseteq 2^{<\omega}$ is a *Sacks tree* if for for every $s \in p$ there is $t \in p$ such that $s \subseteq t$ and t is a *splitting node of* p (i.e., $t^\frown 0, t^\frown 1 \in p$). Since $\dot{x}$ is a new real, $S_p(\dot{x})$ is a Sacks tree. Recall that the set of all $\dot{x}$-nice conditions is a dense set.

Lemma 34   *Let* p *be* $\dot{x}$-nice *and* $n > 0$.

1. *If* $h : n \longrightarrow 2$ *then there is* $q \le_n p$ *such that* $\dot{x}[q[h^\frown 0]]$ *and* $\dot{x}[q[h^\frown 1]]$ *are incompatible (as nodes in the tree* $2^{<\omega}$).
2. *There is* $r \le_n p$ *such that* $\dot{x}[r[h^\frown 0]]$ *and* $\dot{x}[r[h^\frown 1]]$ *are incompatible for every* $h : n \longrightarrow 2$.

Proof   Let $h : n \longrightarrow 2$ and since $S_p(\dot{x})$ is a Sacks tree, there must be $h^\frown 0 \subseteq g_0^0, g_0^1$ and $h^\frown 1 \subseteq g_1^0, g_1^1$ such that both $\dot{x}[p[g_0^0]]$ and $\dot{x}[p[g_0^1]]$ are incompatible and both $\dot{x}[p[g_1^0]]$ and $\dot{x}[p[g_1^1]]$ are incompatible. We may also assume there is m such that

all g_0^0, g_0^1, g_1^0 and g_1^1 have m as its domain. There must be i, j such that $\dot{x}[p[g_0^i]]$ and $\dot{x}[p[g_1^j]]$ are incompatible, with out loosing generality, we may assume $i = j = 0$. We define the condition $q \leq p$ with the following properties:

1. $E_q = E_p \setminus \{ [a_p(l)]_p \mid l \in (n, m) \}$.

2. If $l \in (n, m)$ and $f : 2^{A_p \cap a_p(l)} \longrightarrow 2$ then the following holds:

   (a) If $f(a_p(n)) = 0$ then $H_q^{a_p(l)}(f) = g_0^0(l)$.

   (b) If $f(a_p(n)) = 1$ then $H_q^{a_p(l)}(f) = g_1^0(l)$.

   Note that $q[h^{\smallfrown}0] = p[g_0^0]$ and $q[h^{\smallfrown}1] = p[g_1^0]$, so the result immediately follows. The second part of the lemma is proved by applying iteratively the first part. ∎

We need the following notion:

Definition 35 Let p be an $\dot{x}$-nice condition and $n \in \omega$. We say that p is n-*separating* if for all $h, g : n + 1 \longrightarrow 2$ then $\dot{x}[p[h]]$ and $\dot{x}[p[g]]$ are incompatible whenever $h \neq g$. We say p is ω-*separating* if it is n-separating for every $n \in \omega$.

Note that if p is n-separating and $q \leq_n^* p$ then q is n-separating too (see Lemma 23). For convenience, we will say that every n-nice condition is -1-separating.

Lemma 36 If p is $(n-1)$-separating then $D = \{ r \leq_n^* p \mid r \text{ is } n\text{-separating} \}$ is $\leq_n^*$-open dense below p.

Proof Note that D is $\leq_n^*$-open by the previous remark, we will see it is also $\leq_n^*$-dense below p. Let $q \leq_n^* p$ and by the previous lemma, we can find $r \leq_n q$ (so $r \leq_n^* q$) such that for that $\dot{x}[r[h^{\smallfrown}0]]$ and $\dot{x}[r[h^{\smallfrown}1]]$ are incompatible for every $h : n \longrightarrow 2$. Since p is $(n-1)$-separating, r is n-separating. ∎

We can now conclude the following:

Corollary 37 The set of all ω-separating conditions is dense. Furthermore, if p is n-nice, then there is $q \leq_n^* p$ such that q is ω-separating.

Proof Let $p \in \mathbb{S}(\mathcal{U})$ be an n-nice condition, we will prove p has a ω-separating extension. We recursively construct a 1-fusion sequence $T = \langle q_i, r_i \rangle_{i < \omega}$ with the following properties:

1. q_0 and r_0 are two incompatible extensions of p with $E_{q_0} = E_{r_0}$.

2. Both q_i, r_i are i-separating.

This can be done by Lemmas 17 and 36. It is easy to see that any fusion of T will have the desired properties. ∎

If $s, t \in 2^{<\omega}$ are two incompatible finite sequences, define $\mathsf{c}_{\min}(s, t)$ to be the parity of the length of the biggest initial segment shared by both s and t. Note that this is an abuse of notation since the domain of $\mathsf{c}_{\min}$ is $[2^\omega]^2$. We will now define the following:

Definition 38 Let $p \in \mathbb{S}(\mathcal{U})$, $n \in \omega$ and $i < 2$. The condition p is called (n, i)-*faithful* if p is n-separating and $\mathsf{c}_{\min}(\dot{x}[p[h]], \dot{x}[p[g]]) = i$ for every $h, g \in 2^{n+1}$ such that $h \neq g$.

If p is (n, i)-faithful and $q \leq_n^* p$, then q is also (n, i)-faithful (see Lemma 23). Once again, for convenience, we will say that p is $(-1, i)$-faithful if p is $\dot{x}$-nice. We can finally prove the main result on this section.

Proposition 39 *If* $\mathcal{U}$ *is an ultrafilter, then* $\mathbb{S}(\mathcal{U})$ *preserves* c_{min}*-covering.*

Proof Let $\dot{x}$ be a $\mathbb{S}(\mathcal{U})$-name for a new element of 2^{ω}. Fix a condition p that is ω-separating. Given $n \in \omega, q \leq p$ and $h : n \longrightarrow 2$, let $I_q(h)$ be the union of all the classes $\left[a_q(m)\right]_q$ with the following properties:

1. $n < m$.
2. For every function $g \in 2^m$ with the property that $h \subseteq g$, we have that $c_{min}\left(\dot{x}\left[q\left[g^\frown 0\right]\right], \dot{x}\left[q\left[g^\frown 1\right]\right]\right) = 1$.

We will now proceed by cases, first assume there are $q \leq p$ and $h \in 2^{<\omega}$ such that $I_q(h) \in \mathcal{U}$. Let $r \leq q[h]$ such that $dom(E_r) = I_q(h)$ and every H_r^l is a constant function for $l \in A_q \backslash A_r$. It is then easy to see that $S_r(\dot{x})$ is a 1-monochromatic tree and we are done.

Now, we assume that $I_q(h) \notin \mathcal{U}$ for every $q \leq p$ and $h \in 2^{<\omega}$. Given $q \leq p$ and $n < \omega$ let $F_n(q) = \{r \leq_n^* q \mid r$ is $(n, 0)$-faithful$\}$.

Claim 40 *If* q *is* $(n - 1, 0)$*-faithful then* $F_n(q)$ *is* $\leq_n^*$*-open dense below* q.

It is clear that $F_{n+1}(q)$ is $\leq_n^*$-open below q, we will now prove that it is also $\leq_n^*$-dense. Let $r \leq_n^* q$, we will extend r to a $(n, 0)$-faithful condition. Every $\leq_n^*$-extension of q is n-nice, so by Corollary 37, we may assume that r is ω-separating. Let $2^{n+1} = \{h_i \mid i < k\}$. We know $B = I_r(h_1) \cup \cdots \cup I_r(h_k) \in \mathcal{U}^*$ (since each $I_r(h_i) \in \mathcal{U}^*$ by hypothesis) and since B is the union of E_r-classes, then there is $m > n + 1$ such that $\left[a_r(m)\right]_r \cap B = \emptyset$. Since $a_r(m) \notin I_r(h_i)$, then for every $i < k$, there is $g_i : m \longrightarrow 2$ extending h_i such that $c_{min}\left(\dot{x}\left[r\left[g_i^\frown 0\right]\right], \dot{x}\left[r\left[g_i^\frown 1\right]\right]\right) = 0$ (recall that r is ω-separating, so in particular it is m-nice for every $m \in \omega$). We now define a condition r_1 with the following properties:

1. $r_1 \leq_n r$.
2. $dom(E_{r_1}) = dom(E_r) \backslash \bigcup \{[a_r(l)]_r \mid n < l < m\}$.
3. If $m \leq l$ then $[a_{r_1}(l)]_{r_1} = [a_r(l)]_r$.
4. If $n < l < m$ and $f \in 2^{A_n \cap a_r(l)}$ then $H_{r_1}^{a_r(l)}(f) = g_i(l)$, where $i < k$ is (the unique) such that $f(a_r(j)) = h_i(j)$ for every $j \leq n$.

It is now easy to see that r_1 is $(n, 0)$-faithful, this finishes the proof of the claim.

We now recursively construct a 1-fusion sequence $T = \langle q_i, r_i \rangle_{i < \omega}$ with the following properties:

1. q_0 and r_0 are two incompatible extensions of p with $E_{q_0} = E_{r_0}$.
2. Both q_i and r_i are $(i, 0)$-faithful.

Let $\bar{p}$ be a fusion of T. It follows that $\bar{p}$ is an extension of p and $S_{\bar{p}}(\dot{x})$ is a 0-monochromatic tree. ∎

4 Iterating the Shelah forcing

Our next task is to prove that if we perform a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \omega_2 \rangle$ such that for $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{S}(\dot{\mathcal{U}}_\alpha)$" for all $\alpha < \omega_2$ (where $\dot{\mathcal{U}}_\alpha$ is a $\mathbb{P}_\alpha$-name for an ultrafilter), then $\mathbb{P}_{\omega_2}$ preserves c_{min}-covering. We do not know, in general, if the iteration of forcings that preserve c_{min}-covering preserves c_{min}-covering, but we were able to prove it in our case. In this section, we will prove some technical lemmas

regarding the iteration that will be useful in the following sections. Right now, our purpose is to extend our work with fusion sequences to every $\mathbb{P}_\alpha$. We hope that the material developed in this section will have more applications in the future, since it provides a very convenient way to work with the iteration of forcings of the form $\mathbb{S}(\mathcal{U})$. Moreover, it should be possible to perform a similar analysis for other forcings that do not have an axiom A structure, yet its main properties (like properness, not adding unbounded reals...) are proved using games or trees (the first example that comes to mind is the Grigorieff forcing of a P-point).

For the following sections, fix a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \omega_2 \rangle$ as described in the beginning of the section.

Definition 41   Let $\alpha \leq \omega_2, F \in [\alpha]^{<\omega}, \eta : F \longrightarrow \omega$ and $p, q \in \mathbb{P}_\alpha$.

1. Define $p \leq_{F,\eta} q$ if $p \leq q$ and for every $\beta \in F$ it is the case that $p \upharpoonright \beta \Vdash$ "$p(\beta) \leq_{\eta(\beta)} q(\beta)$."

2. $p \leq_{F,\eta}^* q$ if $p \leq q$ and for every $\beta \in F$ it is the case that $p \upharpoonright \beta \Vdash$ "$p(\beta) \leq_{\eta(\beta)}^* q(\beta)$."

3. If $\sigma \in \prod\limits_{\beta \in F} 2^{\eta(\beta)}$, we define the condition $p * \sigma \in \mathbb{P}_\alpha$ as follows:
   (a) If $\delta \notin F$ then $(p * \sigma) \upharpoonright \delta \Vdash$ "$(p * \sigma)(\delta) = p(\delta)$."
   (b) If $\delta \in F$ then $(p * \sigma) \upharpoonright \delta \Vdash$ "$(p * \sigma)(\delta) = p(\delta)[\sigma(\delta)]$" (recall that $\sigma(\delta) \in 2^{\eta(\delta)}$).

In a similar way as before, define the following:

Definition 42   Let $\alpha \leq \omega_2, F \in [\alpha]^{<\omega}, \eta : F \longrightarrow \omega, p \in \mathbb{P}_\alpha$ and $D \subseteq \mathbb{P}_\alpha$.

1. D is $(F, \eta)^*$-*dense below* p if for every $q \leq_{F,\eta}^* p$ there is $r \leq_{F,\eta}^* q$ such that $r \in D$.

2. D is $(F, \eta)^*$-*open below* p if for every r such that $r \leq_{F,\eta}^* p$ and $r \in D$ then $q \in D$ whenever $q \leq_{F,\eta}^* r$.

3. D is $(F, \eta)^*$-*open dense below* p if it is both $(F, \eta)^*$-open and $(F, \eta)^*$-dense below p.

4. D is (F, η)-*dense below* p if for every $q \leq_{F,\eta} p$ there is $r \leq_{F,\eta} q$ such that $r \in D$.

5. D is (F, η)-*open below* p if for every r such that $r \leq_{F,\eta} p$ and $r \in D$ then $q \in D$ whenever $q \leq_{F,\eta} r$.

6. D is (F, η)-*open dense below* p if it is both (F, η)-open and (F, η)-dense below p.

We use similar conventions as before. The expression "D is (F, η)-dense" actually means that D is (F, η)-dense below the largest condition (similarly for the other notions). The expression "D is (F, η)-open and (F_1, η_1)-dense below p" is a shorthand for "D is (F, η)-open below p and (F_1, η_1)-dense below p." Note that if $F \subseteq F_1, \eta \leq \eta_1 \upharpoonright F_1$ and D is (F, η)-open below p, then D is (F_1, η_1)-open below p.

Let $\eta : F \longrightarrow \omega$, by $\eta + 1 : F \longrightarrow \omega$ we denote the function given by $(\eta + 1)(\alpha) = \eta(\alpha) + 1$. For $\eta_1, \eta_2 \in \omega^F$, define $\eta_1 \leq \eta_2$ if $\eta_1(\alpha) \leq \eta_2(\alpha)$ for every $\alpha \in F$.

The following is the generalization of Lemma 23:

Lemma 43   Let $\alpha \leq \omega_2, F \in [\alpha]^{<\omega}, \eta : F \longrightarrow \omega, p, q \in \mathbb{P}_\alpha$ such that $q \leq_{F,\eta}^* p$. If $\sigma \in \prod\limits_{\beta \in F} 2^{\eta(\beta)+1}$, then $q * \sigma \leq p * \sigma$.

If $D \subseteq \mathbb{P}_\alpha$ is an open dense set and $F \in [\alpha]^{<\omega}, \eta : F \longrightarrow \alpha$, define $\widetilde{D}_{F,\eta} = \{p \in \mathbb{P}_\alpha \mid \forall \sigma \in \prod\limits_{\delta \in F} 2^{\eta(\delta)} (p * \sigma \in D)\}$.

Lemma 44 *Let* $D \subseteq \mathbb{P}_\alpha$ *be an open dense set,* $F \in [\alpha]^{<\omega}$ *and* $\eta : F \longrightarrow \alpha$. *The set* $\widetilde{D}_{F,\eta} = \{r \in \mathbb{P}_\alpha \mid \forall \sigma \in \prod_{\delta \in F} 2^{\eta(\delta)} \ (r * \sigma \in D)\}$ *is* (F, η)*-open dense.*

Proof We will first prove that $\widetilde{D}_{F,\eta}$ is (F, η)-dense. Let p be a condition, enumerate $\prod_{\delta \in F} 2^{\eta(\delta)} = \{\sigma_i \mid i \leq k\}$. Recursively, we build $\{p_i \mid i \leq k\}$ such that the following holds:

1. $p_0 \leq_{F,\eta} p$.
2. $p_{i+1} \leq_{F,\eta} p_i$ for $i + 1 \leq k$.
3. $p_i * \sigma_i \in D$.

This is easy to do. It follows that $p_k \in D$ and $p_k \leq_{F,\eta} p$. It is easy to see that $\widetilde{D}_{F,\eta}$ is (F, η)-open. ∎

We will recall a well-known forcing lemma that will be frequently used (for a proof, see Lemma 1.19 in the first chapter of the book [14]):

Lemma 45 (Definition by cases) *Let* $\mathbb{P}$ *be a partial order,* $A = \{p_\alpha \mid \alpha \in \kappa\} \subseteq \mathbb{P}$ *an antichain and* $\{\dot{x}_\alpha \mid \alpha \in \kappa\}$ *be a set of* $\mathbb{P}$*-names. There is a* $\mathbb{P}$*-name* $\dot{y}$ *such that* $p_\alpha \Vdash$ *"* $\dot{y} = \dot{x}_\alpha$*" for every* $\alpha \in \kappa$.

The following technical notion will be frequently used:

Definition 46 Let M be a countable elementary submodel of $\mathsf{H}(\kappa)$ (for some big enough regular κ) and $\alpha \in M$ an ordinal. We say that $L = \{(F_n, \eta_n) \mid n \in \omega\}$ is *suitable for* (M, α) if the following holds for every $n \in \omega$:

1. F_n is a finite subset of $M \cap \alpha$.
2. $F_n \subseteq F_{n+1}$.
3. $|F_n| \leq n$ and $|F_{n+1}| \leq |F_n| + 1$.
4. $\bigcup_{n \in \omega} F_n = M \cap \alpha$.
5. $\eta_n : F_n \longrightarrow \omega$.
6. $\eta_n \leq \eta_{n+1} \restriction F_n$
7. If $\beta \in M \cap \alpha$ then $\langle \eta_n(\beta) \rangle_{n \in \omega} \longrightarrow \infty$.

Note that if $L = \{(F_n, \eta_n) \mid n \in \omega\}$ is suitable for (M, α), then $F_n, \eta_n \in M$ for every $n \in \omega$ (however, it may be the case that $L \notin M$). The following lemma is easy and left to the reader:

Lemma 47 *Let* M *be a countable elementary submodel of* $\mathsf{H}(\kappa)$, $\alpha \in M$ *and* $L = \{(F_n, \eta_n) \mid n \in \omega\}$ *suitable for* (M, α). *Let* $\beta \in M \cap \alpha$, *define* $F'_n = F_n \cap \beta, \eta'_n = \eta_n \restriction F'_n$ *and* $L' = \{(F'_n, \eta'_n) \mid n \in \omega\}$. *Then* L' *is suitable for* (M, β).

The following is the generalization of a 1-fusion sequence:

Definition 48 Let M be a countable elementary submodel of $\mathsf{H}(\kappa)$ (for some big enough regular κ) such that $\mathbb{P}_\alpha \in M$ and $L = \{(F_n, \eta_n) \mid n \in \omega\}$ be suitable for (M, α). We say that a tree $T \subseteq (\mathbb{P}_\alpha \cap M)^{<\omega}$ (ordered by end-extension) is an *α-fusion tree*

(with respect to M and L) if there are $d_-^T, w_-^T : T \longrightarrow \mathbb{P}_\alpha$ and $\langle \dot{e}_n^T, \dot{u}_n^T \rangle_{n \in \omega}$ a sequence of $\mathbb{P}_\alpha$-names such that the following conditions hold:

1. $|T_0| = 1$ and if $s = \langle p_0, \dots, p_n \rangle \in T$, then $p_{m+1} \leq_{F_m, \eta_m}^* p_m$ for every $m < n$.
2. If $s \in T$, then $suc_T(s) = \{ d_s^T, w_s^T \}$.
3. If $F_{n+1} = F_n$ and $s \in T_n$, then $d_s^T = w_s^T$ (so s has only one immediate successor in T).
4. Let $n \in \omega$ such that $F_{n+1} \backslash F_n \neq \emptyset$. Pick γ the (unique) element of $F_{n+1} \backslash F_n$ and $s = \langle p_0, \dots, p_n \rangle \in T_n$. The following conditions hold:
   (a) $suc_T(s) = \{ d_s^T, w_s^T \}$.
   (b) $d_s^T \upharpoonright \gamma = w_s^T \upharpoonright \gamma$.
   (c) $d_s^T \upharpoonright \gamma \Vdash$ "$\dot{d}_s^T(\gamma) = \dot{e}_{n^T}$" and $w_s^T \upharpoonright \gamma \Vdash$ "$\dot{w}_s^T(\gamma) = \dot{u}_n^T$."
   (d) $d_s^T \upharpoonright \gamma \, (= w_s^T \upharpoonright \gamma)$ forces that $\dot{e}_n^T, \dot{u}_n^T$ are incompatible elements of $\mathbb{S}(\dot{\mathcal{U}}_\gamma)$ but $E_{\dot{e}_n^T} = E_{\dot{u}_n^T}$.
5. Assume $s = \langle p_0, \dots, p_n \rangle, t = \langle p_0', \dots, p_n' \rangle, m = \triangle(s, t)$ and $\gamma \in F_{m+1} \backslash F_m$. If $p_n \upharpoonright \gamma = p_n' \upharpoonright \gamma$ the following holds:
   (a) In case $F_n \neq F_{n+1}$ the following holds:
      i. $d_s^T \upharpoonright \gamma = d_t^T \upharpoonright \gamma$ and $w_s^T \upharpoonright \gamma = w_t^T \upharpoonright \gamma$.
      ii. $d_s^T \upharpoonright \gamma \Vdash$ "$\langle p_n(\gamma), p_n'(\gamma), d_s^T(\gamma), d_t^T(\gamma) \rangle$ is an $\eta_n(\gamma)$-nice sequence."
      iii. $w_s^T \upharpoonright \gamma \Vdash$ "$\langle p_n(\gamma), p_n'(\gamma), w_s^T(\gamma), w_t^T(\gamma) \rangle$ is an $\eta_n(\gamma)$-nice sequence."
   (b) In case $F_n = F_{n+1}$, we have the following:
      i. $d_s^T \upharpoonright \gamma = d_t^T \upharpoonright \gamma$.
      ii. $d_s^T \upharpoonright \gamma \Vdash$ "$\langle p_n(\gamma), p_n'(\gamma), d_s^T(\gamma), d_t^T(\gamma) \rangle$ is an $\eta_n(\gamma)$-nice sequence."

Since this definition is crucial for the paper, we find it convenient to make the following clarifications:

1. We view $d_-^T, w_-^T : T \longrightarrow \mathbb{P}_\alpha$ as operators that help us define the immediate successors of a node of T. In this way, if $s \in T_n$, then the immediate successors of s are $s \frown d_s^T$ and $s \frown w_s^T$. We also have that $s \frown d_s^T = s \frown w_s^T$ (so s has only one immediate successor) if and only if $F_{n+1} = F_n$.
2. Let $n \in \omega$ and γ are such that $\gamma \in F_{n+1} \backslash F_n$. If $s \in T_n$, then $\dot{d}_s^T(\gamma)$ is forced to be $\dot{e}_s^T$ and $\dot{w}_s^T(\gamma)$ is forced to be $\dot{u}_n^T$. Note that $\dot{e}_n^T$ and $\dot{u}_n^T$ depend only on n and not in s. In this way, if t is also in T_n, then $\dot{d}_t^T(\gamma)$ is also forced to be $\dot{e}_t^T$ and $\dot{w}_t^T(\gamma)$ is also forced to be $\dot{u}_n^T$. However, since $\dot{e}_n^T$ and $\dot{w}_s^T(\gamma)$ are names and not "real objects," it might be the case that $\dot{d}_s^T \upharpoonright \gamma$ and $\dot{d}_t^T \upharpoonright \gamma$ evaluate $\dot{e}_n^T$ in very different ways (the same for $\dot{w}_s^T \upharpoonright \gamma$ and $\dot{w}_t^T \upharpoonright \gamma$).
3. By "T is a fusion tree," we mean that T is an α-fusion tree for some α.
4. In case $F_{n+1} = F_n$, we have no use for $\dot{e}_n^T$ and $\dot{u}_n^T$. In the definition of α-fusion tree, we could have asked that $\dot{e}_n^T$ and $\dot{u}_n^T$ are only defined for the $n \in \omega$ such that $F_{n+1} \neq F_n$. We did not do it this way because we thought it would only add extra complexity to the (already quite complex) definition. In the paper, when we are building fusion trees, we will simply leave this case undefined.
5. In the definition of fusion tree, we demanded that $T \subseteq M$, so the operators d_-^T and w_-^T must always take values in M and the names $\dot{e}_n^T$ and $\dot{u}_n^T$ must always be in M.

6. If T is a fusion tree, the operators will always be denoted by d_-^T and w_-^T and the sequence of names will always be $\dot{e}_n^T$ and $\dot{u}_n^T$ for $n \in \omega$. However, if there is no risk of confusion, we will simply write $d_s, w_s, \dot{e}_n,$ and $\dot{u}_n$ instead of $d_s^T, w_s^T, \dot{e}_n^T,$ and $\dot{u}_n^T$.

7. For the convenience of the reader, whenever there is a fusion tree around, every occurrence of $d, w, e,$ and u (with either subindices or superinduces) refer to the operators of a fusion tree.

The definition of 1-fusion sequence from the previous section is essentially the same as the one of a 1-fusion tree, the only difference is the presence for the elementary submodel (in fact, the role of M is simply to help us with bookkeeping arguments, but it really could be avoided if we wanted). If a tree $T \subseteq (\mathbb{P}_\alpha \cap M)^{<\omega}$ of height k satisfy all the above properties but only for $n \leq k$, we will say that T is an *α-finite fusion tree of height k*.

Lemma 49 *Let M be a countable elementary submodel of $\mathsf{H}(\kappa)$, $\alpha \in M, L = \{(F_n, \eta_n) \mid n \in \omega\}$ suitable for (M, α) and $T \subseteq (\mathbb{P}_\alpha \cap M)^{<\omega}$ an α-fusion tree (with respect to M and L). Let $n \in \omega$ and $s = \langle p_0, \ldots, p_{n+1} \rangle, t = \langle q_0, \ldots, q_{n+1} \rangle \in T_{n+1}$. If $s \neq t$, then p_{n+1} and q_{n+1} are incompatible.*

Proof Let $m = \triangle(s, t)$, by point 4 of definition 48, we get that p_m and q_m are incompatible. Since $p_{n+1} \leq p_m$ and $q_{n+1} \leq q_m$, the result follows. ∎

By the above lemma, given $T \subseteq (\mathbb{P}_\alpha \cap M)^{<\omega}$ an α-fusion tree, we could identify each $s = \langle p_0, \ldots, p_n \rangle \in T$ with its last element. There is no risk of confusion by the lemma. Using this identification, the order of the tree corresponds with the order of $\mathbb{P}_\alpha$. In this section, we will not use the identification, but it will be convenient in later sections when the notation becomes even more involved (we will warn the reader when we use this abuse of notation).

In order to use inductive arguments, we need to restrict an α-fusion tree to a lesser ordinal. The objective of the following definitions and results is to prove that the restriction of a fusion tree is a fusion tree.

Definition 50 Let T be an α-fusion tree and $\beta < \alpha$. Define $\sim_\beta$ a relation on T as follows: given $s = (p_0^s, \ldots, p_n^s)$ and $t = (p_0^t, \ldots, p_n^t)$, define $s \sim_\beta t$ if the following conditions hold:

1. $n = m$.
2. $p_i^s \upharpoonright \beta = p_i^t \upharpoonright \beta$ for every $i \leq n$.

It is easy to see that $\sim_\beta$ is an equivalence relation on T. We now have the following:

Lemma 51 *Let M be a countable elementary submodel of $\mathsf{H}(\kappa)$, $\alpha \in M$ and $L = \{(F_n, \eta_n) \mid n \in \omega\}$ suitable for (M, α). Let $\beta \in M \cap \alpha$, define $F_i' = F_i \cap \beta$ for every $i \in \omega$. Let $n \in \omega$ and $s, z \in T_n$ such that $s \sim_\beta z$. The following holds:*

1. $d_s \upharpoonright \beta = d_z \upharpoonright \beta$ and $w_s \upharpoonright \beta = w_z \upharpoonright \beta$.
2. $(s^\frown d_s) \sim_\beta (z^\frown d_z)$ and $(s^\frown w_s) \sim_\beta (z^\frown w_z)$.
3. *If $F_{n+1}' \backslash F_n' \neq \emptyset$, then $d_s \upharpoonright \beta \neq w_s \upharpoonright \beta$.*
4. *If $F_{n+1}' \backslash F_n' = \emptyset$, then $d_s \upharpoonright \beta = w_s \upharpoonright \beta$.*

Proof We will first prove point 1. Denote by A the $\sim_\beta$-equivalence class of s (and of z as well, obviously). Let $a = (p_0^a, \ldots, p_n^a)$ and $b = (p_0^b, \ldots, p_n^b)$ be two elements of A

and $\xi < \alpha$. We say that ξ is a *disagreement point of a and b* if the following conditions hold:

1. There is $m \leq n$ such that $\xi \in F_{m+1} \backslash F_m$.
2. Denote $\overline{a} = (p_0^a, \ldots, p_{m-1}^a)$ and $\overline{b} = (p_0^b, \ldots, p_{m-1}^b)$. Either $p_m^a = d_{\overline{a}}$ and $p_m^b = w_{\overline{b}}$ or $p_m^a = w_{\overline{a}}$ and $p_m^b = d_{\overline{b}}$.

Note that if ξ is a disagreement point of a and b, then $\beta \leq \xi$ (this is because $a \sim_\beta b$). Denote by $dis(a, b)$ the set of all disagreement points of a and b. The following remarks are easy to see:

1. $dis(a, b)$ is a finite set and $\beta \cap dis(a, b) = \emptyset$.
2. $dis(a, b) = \emptyset$ if and only if $a = b$.
3. If $\triangle(a, b) = m$ and $\xi \in F_{m+1} \backslash F_m$, then $\xi \in dis(a, b)$.

We will say that a and b are *near* if $dis(a, b)$ has at most one element. We will now prove the following:

Claim 52   *Let $a, b \in A$. If a and b are near, then $d_a \restriction \beta = d_b \restriction \beta$ and $w_a \restriction \beta = w_b \restriction \beta$.*

Let $a = (p_0^a, \ldots, p_n^a)$ and $b = (p_0^b, \ldots, p_n^b)$. If $a = b$, the claim is trivially true, so assume that $a \neq b$. Let $\xi \in dis(a, b)$ and $m \in \omega$ such that $\xi \in F_{m+1} \backslash F_m$. We claim that $a \sim_\xi b$ (in other words, $p_i^a \restriction \xi = p_i^b \restriction \xi$ for every $i \leq n$).

1. If $i < m$, then $p_i^a = p_i^b$ (so $p_i^a \restriction \xi = p_i^b \restriction \xi$).
2. If $i = m$, then $p_i^a \restriction \xi = p_i^b \restriction \xi$ by point 5(a) of Definition 48.
3. If $i > m$, then $p_i^a \restriction \xi = p_i^b \restriction \xi$ because T is fusion tree and ξ is the only point of disagreement between a and b.

We will provide more details for the last point. We proceed by induction over i. Assume that $p_i^a \restriction \xi = p_i^b \restriction \xi$ for $m \leq i < n$, we will prove that $p_{i+1}^a \restriction \xi = p_{i+1}^b \restriction \xi$. In case $F_{i+1} = F_i$, the conclusion follows by point 5(b) of Definition 48. Now assume that $F_{i+1} \neq F_i$ and let $\delta \in F_{i+1} \backslash F_i$. Since δ is not a disagreement point, by point 5(a) of Definition 48, we conclude that $p_{i+1}^a \restriction \xi = p_{i+1}^b \restriction \xi$.

In particular, we get that $p_n^a \restriction \xi = p_n^b \restriction \xi$. By point 5 of Definition 48, we get that $d_a \restriction \xi = d_b \restriction \xi$ and $w_a \restriction \xi = w_b \restriction \xi$. Since $\beta \leq \xi$, we conclude that $d_a \restriction \beta = d_b \restriction \beta$ and $w_a \restriction \beta = w_b \restriction \beta$. This finishes the proof of the claim.

Claim 53   *Let $a, b \in A$ with $a \neq b$. There is $c \in A$ such that the following holds:*

1. *a and c are near.*
2. *$|dis(c, b)| < |dis(a, b)|$.*

Let $a = (p_0^a, \ldots, p_n^a), b = (p_0^b, \ldots, p_n^b), m = \triangle(a, b)$, and $\xi \in F_{m+1} \backslash F_m$. Define $c = (p_0^c, \ldots, p_n^c)$ as follows:

1. $p_i^c = p_i^a$ for $i < m$ (so $p_i^c = p_i^b$).
2. $p_m^c = p_m^b$.
3. Let $i \geq m$, denote $\overline{a} = (p_0^a, \ldots, p_i^a)$ and $\overline{c} = (p_0^c, \ldots, p_i^c)$. Let $p_{i+1}^c = d_{\overline{c}}$ if $p_{i+1}^a = d_{\overline{a}}$ and $p_{i+1}^c = w_{\overline{c}}$ if $p_{i+1}^a = w_{\overline{a}}$.

Note that c is in the tree T since we are always taking the successor of a node in T. It follows by the construction that $\triangle(a, c) = m$ and by point 5 of Definition 48, we get that $p_i^c \restriction \xi = p_i^a \restriction \xi$ for all $i \leq n$, hence $c \in A$. It is clear that a and c are near and $|dis(c, b)| = |dis(a, b)| - 1$. This finishes the proof of the claim.

We can now finish the proof of the first point of the lemma. Recall that $s \sim_\beta z$ and we wanted to prove that $d_s \upharpoonright \beta = d_z \upharpoonright \beta$ and $w_s \upharpoonright \beta = w_z \upharpoonright \beta$. Now, by claim 53, we can find a sequence $(a_0, \ldots, a_k)$ such that for every $i \leq k$, the following conditions hold:

1. $a_i \in A$.
2. $a_0 = s$ and $a_k = z$.
3. a_{i+1} is near of a_i.

By Claim 52, we conclude that $d_s \upharpoonright \beta = d_z \upharpoonright \beta$ and $w_s \upharpoonright \beta = w_z \upharpoonright \beta$.

The second point of the lemma follows by the first one, we will now prove the third point. Assume that $F'_{n+1} \backslash F'_n \neq \emptyset$, we want to prove that $d_s \upharpoonright \beta \neq w_s \upharpoonright \beta$. Let $\gamma \in F'_{n+1} \backslash F'_n$. Since $F'_{n+1} = F_{n+1} \cap \beta$ and $F'_n = F_n \cap \beta$, it follows that $\gamma < \beta$. By point 4 of definition 48, we know that d_s and w_s differ in the γ component, which entails that $d_s \upharpoonright \beta \neq w_s \upharpoonright \beta$.

We now prove point 4 of the lemma. Assume that $F'_{n+1} \backslash F'_n = \emptyset$, we want to prove that $d_s \upharpoonright \beta = w_s \upharpoonright \beta$. If $F_{n+1} \backslash F_n = \emptyset$, it follows by point 3 of Definition 48 that $d_s = w_s$, so $d_s \upharpoonright \beta = w_s \upharpoonright \beta$. Assume now that $F_{n+1} \backslash F_n \neq \emptyset$. Let $\gamma \in F_{n+1} \backslash F_n$. Since $F'_{n+1} \backslash F'_n = \emptyset$, it must be the case that $\beta \leq \gamma$. By point 5 of Definition 48 we know that $d_s \upharpoonright \gamma = w_s \upharpoonright \gamma$, which implies that $d_s \upharpoonright \beta = w_s \upharpoonright \beta$. ∎

We can now make the following definition:

Definition 54 Let M be a countable elementary submodel of $H(\kappa)$ (for some big enough regular cardinal κ) such that $\mathbb{P}_\alpha \in M$, $L = \{(F_n, \eta_n) \mid n \in \omega\}$ suitable for (M, α) and $T \subseteq (\mathbb{P}_\alpha \cap M)^{<\omega}$ an α-fusion tree. Let $\beta < \alpha$ with $\beta \in M$.

1. Define $T \upharpoonright \beta$ as the tree in $\mathbb{P}_\beta^{<\omega}$ obtained by restricting every condition of T to β.
   In other words, if $s = \langle p_0, \ldots, p_n \rangle \in T$, then $s' = \langle p_0 \upharpoonright \beta, \ldots, p_n \upharpoonright \beta \rangle \in T \upharpoonright \beta$.
2. Let $s = \langle p_0, \ldots, p_n \rangle \in T$ and $z = \langle p_0 \upharpoonright \beta, \ldots, p_n \upharpoonright \beta \rangle$. Define $d_z^{T \upharpoonright \beta} = d_s^T \upharpoonright \beta$ and $w_z^{T \upharpoonright \beta} = w_s^T \upharpoonright \beta$.
3. For every $n \in \omega$ such that $(F_{n+1} \backslash F_n) \cap \beta \neq \emptyset$, let $\dot{e}_n^{T \upharpoonright \beta} = \dot{e}_n^T$ and $\dot{w}_n^{T \upharpoonright \beta} = \dot{w}_n^T$

The operators $d^{T \upharpoonright \beta}$ and $w^{T \upharpoonright \beta}$ are well defined by Lemma 51. We need that the restriction of a fusion tree is a fusion tree. This is the content of the following proposition.

Proposition 55 *Let M be a countable elementary submodel of $H(\kappa)$, $\alpha \in M$ and $L = \{(F_n, \eta_n) \mid n \in \omega\}$ suitable for (M, α). Let $\beta \in M \cap \alpha$, define $F'_n = F_n \cap \beta$, $\eta'_n = \eta_n \upharpoonright F'_n$ and $L' = \{(F'_n, \eta'_n) \mid n \in \omega\}$. If $T \subseteq (\mathbb{P}_\alpha \cap M)^{<\omega}$ is an α-fusion tree (with respect to M and L), then $T \upharpoonright \beta \subseteq (\mathbb{P}_\beta \cap M)^{<\omega}$ is a β-fusion tree (with respect to M and L').*

Proof We already noted that L' is suitable for (M, β). Let $S = T \upharpoonright \beta$. For this proof, we adopt the following convention: if $s = \langle q_0, \ldots, q_n \rangle \in T$, define $s' = \langle q_0 \upharpoonright \beta, \ldots, q_n \upharpoonright \beta \rangle$ (we are not assuming this convention outside this proof, unless when we mention it explicitly). Note that $S_n = \{s' \mid s \in T_n\}$ for every $n \in \omega$. We will prove that S is a β-fusion tree. The first point from Definition 48 is clear.

Note that for every $s \in S$ there may be many $t \in T$ such that $t' = s$. This is where Lemma 51 comes into play. It is easy to see that $suc_S(s) = \{x \upharpoonright \beta \mid \exists t \in T (t' = s \wedge x \in suc_T(t))\}$. Point 2 from Definition 48 follows by point 1 of Lemma 51. Point 3 of the definition of fusion tree follows by point 4 of Lemma 51. Point 4 follows from the definition of $T \upharpoonright \beta$ and Lemma 51.

We will now prove point 5 of the definition of fusion tree. Let $s = \langle p_0^s, \ldots, p_n^s \rangle$ and $t = \langle p_0^t, \ldots, p_n^t \rangle$ two nodes in S, $m = \triangle(s, t)$, $\gamma \in F'_{m+1} \backslash F'_m$ such that $p_n^s \upharpoonright \gamma = p_n^t \upharpoonright \gamma$. now we need the following:

Claim 56 *There are $a = \langle p_0^a, \ldots, p_n^a \rangle$ and $b = \langle p_0^b, \ldots, p_n^b \rangle$ nodes in T such that:*

1. $a' = s$ and $b' = t$.
2. $m = \triangle(a, b)$.

We proceed to prove the claim. First, note that if $a' = s$ and $b' = t$, then $\triangle(a, b) \le m$. We need to prove that we can realize the value m. In order to do that, it is enough to prove the following: If a, b are such that $a' = s$ and $b' = t$ and $m_1 = \triangle(a, b) < m$, then there are a_1, b_1 with the following properties:

1. $a_1' = s$.
2. $b_1' = t$.
3. $m_1 < \triangle(a_1, b_1)$.

Let $\xi \in F_{m_1+1} \backslash F_{m_1}$. Since $m_1 < m$, it must be the case that $\beta \le \xi$. We now define $c = \langle p_0^c, \ldots, p_n^c \rangle$ as follows:

1. If $i < m_1$, then $p_i^c = p_i^b$ (so $p_i^c = p_i^a$ as well).
2. If $i = m_1$, then $p_{m_1}^c = p_{m_1}^b$.
3. If $i > m$, denote $\overline{a} = (p_0^a, \ldots, p_i^a)$ and $\overline{c} = (p_0^c, \ldots, p_i^c)$. Let $p_{i+1}^c = d_{\overline{c}}$ if $p_{i+1}^a = d_{\overline{a}}$ and $p_{i+1}^c = w_{\overline{c}}$ if $p_{i+1}^a = w_{\overline{a}}$.

It follows that $c' = a'$ (since $\beta \le \xi$ and by point 5 of the definition of fusion tree) and $m_1 < \triangle(c, b)$, so c and b have the desired properties. This finishes the proof of the claim.

We now return to prove that S satisfies point 5 of the definition of fusion tree. Using the claim above, we pick $a = \langle p_0^a, \ldots, p_n^a \rangle$ and $b = \langle p_0^b, \ldots, p_n^b \rangle$ such that $a' = s$, $b' = t$ and $m = \triangle(a, b)$. Since $p_n^s \upharpoonright \gamma = p_n^t \upharpoonright \gamma$, it follows that $p_n^a \upharpoonright \gamma = p_n^b \upharpoonright \gamma$. Recall that $d_s^S = d_a^T \upharpoonright \beta$, $w_s^S = w_a^T \upharpoonright \beta$ and $d_t^S = d_b^T \upharpoonright \beta$, $w_t^S = w_b^T \upharpoonright \beta$. We need to proceed by cases:

Case 57 $F'_{n+1} \ne F'_n$.

In particular, we get that $F_{n+1} \ne F_n$. Since T is a fusion tree (in particular, it satisfies point 5(a) of Definition 48), we get the desired conclusion.

Case 58 $F'_{n+1} = F'_n$ and $F_{n+1} = F_n$.

In this case, we use that T satisfies point 5(b) of Definition 48 and we are done.

Case 59 $F'_{n+1} = F'_n$ and $F_{n+1} \ne F_n$.

For this case, we must prove that $d_s^S \upharpoonright \gamma = d_t^S \upharpoonright \gamma$. Since T satisfies point 5(a) of Definition 48, we are fine in this case.

This finishes the proof that S is a fusion tree. ∎

We will need the following lemma later on.

Lemma 60 *Let M be a countable elementary submodel of $\mathsf{H}(\kappa)$, $\beta + 1 \in M$ and $L = \{(F_n, \eta_n) \mid n \in \omega\}$ suitable for $(M, \beta + 1)$. Let T be a $(\beta + 1)$-fusion tree and $n \in \omega$ the first such that $\beta \in F_{n+1}$. Let $m > n$ and $s = \langle p_0^s, \ldots, p_m^s \rangle \in T_m$. There is a unique $t = \langle p_0^t, \ldots, p_m^t \rangle \in T_m$ such that $\triangle(s, t) = n$ and $s \sim_\beta t$.*

Proof   Define $t = \langle p_0^t, \ldots, p_m^t \rangle$ as follows:

1. If $i < n$, then $p_i^t = p_i^s$.
2. If $i = n$, let $a = \langle p_0^s, \ldots, p_{n-1}^s \rangle$ (so $a = \langle p_0^t, \ldots, p_{n-1}^t \rangle$). Let $p_n^t \in suc_T(a)$ such that $p_n^t \neq p_n^s$.
3. For $i > n$, let $\bar{s} = \langle p_0^s, \ldots, p_{i-1}^s \rangle$ and $\bar{t} = \langle p_0^t, \ldots, p_{i-1}^t \rangle$. Define $p_i^t = d_{\bar{t}}$ if $p_i^s = d_{\bar{s}}$ and $p_i^t = w_{\bar{t}}$ if $p_i^s = w_{\bar{s}}$.

We clearly have that $\triangle(s, t) = n$ and $s \sim_\beta t$. We will now prove that there cannot be more that one. Assume that there is $a = \langle p_1^a, \ldots, p_m^a \rangle \in T_m$ with the following properties:

1. $a \neq t$ and $a \neq s$.
2. $\triangle(a, s) = \triangle(t, s) = n$.
3. $p_i^a \upharpoonright \beta = p_i^s \upharpoonright \beta = p_i^t \upharpoonright \beta$ for every $i \leq m$.

By the above properties, we must have that $n < \triangle(a, t)$. Let $k = \triangle(a, t)$ and $\xi \in F_{k+1} \backslash F_k$. Since $n < k$, we get that $\xi \neq \beta$, so $\xi < \beta$. We will now have that $p_{k+1}^a \upharpoonright (\xi + 1) \neq p_{k+1}^s \upharpoonright (\xi + 1)$, so $p_{k+1}^a \upharpoonright \beta \neq p_{k+1}^s \upharpoonright \beta$, which is a contradiction.   ∎

The following lemma will play a key role in our work:

Definition 61   Let T be an α-fusion tree and $q \in \mathbb{P}_\alpha$. We say q *is compatible with* T if there is a $\mathbb{P}_\alpha$-name for a (possibly new) branch $\dot{R}$ of T such that $q \Vdash$ "$R \subseteq \dot{G}$" (where $\dot{G}$ is the name for the $\mathbb{P}_\alpha$-generic filter).

In the above definition, we are identifying a branch with its image. The following result was inspired by the "properness iteration lemma" used to prove that the iteration of proper forcing is proper (see [1, 14]).

Lemma 62   Let M be a countable elementary submodel of $H(\kappa)$, $\alpha, \beta \in M$ with $\beta < \alpha$ and T an α-fusion tree. Assume there is $q \in \mathbb{P}_\beta$ that is compatible with $T \upharpoonright \beta$. There is $\dot{r}$ such that $q^\frown \dot{r} \in \mathbb{P}_\alpha$ and $q^\frown \dot{r}$ is compatible with T. Moreover, if $\dot{R}$ is a $\mathbb{P}_\beta$-name for a branch through $T \upharpoonright \beta$ such that $q \Vdash$ "$\dot{R} \subseteq \dot{G}_\beta$," then there is $\dot{R}'$ a $\mathbb{P}_\alpha$-name for a branch through T such that $q^\frown \dot{r} \Vdash$ "$\dot{R}' \subseteq \dot{G}_\alpha$" and $q^\frown \dot{r} \Vdash$ "$\dot{R}' \upharpoonright \beta = \dot{R}$"($\dot{G}_\beta$ is the $\mathbb{P}_\beta$-name for the generic filter of $\mathbb{P}_\beta$ and $\dot{G}_\alpha$ is the $\mathbb{P}_\alpha$-name for the generic filter of $\mathbb{P}_\alpha$).

Proof   We prove it by induction on α, the case $\alpha = 1$ follows from the argument of Lemma 20. Now assume $\alpha = \delta + 1$ and the lemma holds for δ. We first tackle the case of $\beta = \delta$.

Let $G \subseteq \mathbb{P}_\beta$ be a generic filter with $q \in G$. Find n the first natural number such that $\beta \in F_n$. For the moment, we work in $V[G]$. In $V[G]$ define the following set:

$$H = \{s \in T \mid s \upharpoonright \beta \in \dot{R}[G] \wedge |s| > n\}.$$

In other words, for every $s = \langle p_0, \ldots, p_l \rangle \in T$ we have that $s \in H$ if and only if $l \geq n$ and $\langle p_0 \upharpoonright \beta, \ldots, p_l \upharpoonright \beta \rangle$ is an initial segment of the branch $\dot{R}[G]$.

Claim 63   If $m > n$, then $|T_m \cap H| = 2$ (in $V[G]$).

We prove the claim. Since $\dot{R}[G]$ is a branch, it follows that $T_m \cap H$ has at least one element. Let $s = \langle p_0, \ldots, p_m \rangle \in T_m \cap H$, by Lemma 60, there is a unique $t = \langle q_0, \ldots, q_m \rangle \in T_m$ such that $\triangle(s, t) = n$ and $q_i \upharpoonright \beta = p_i \upharpoonright \beta$ for every $i \leq m$. It follows

that $t \in T_m \cap H$, so $T_m \cap H$ has at least two elements. Furthermore, if there was $u = \langle r_0, \ldots, r_m \rangle \in T_m \cap H$ with $u \neq s$, we must have that $\triangle(s, u) = n$ (because if $k = \triangle(s, u) \neq n$, then $r_k \upharpoonright \beta$ would be incompatible with $p_k \upharpoonright \beta$ by point 4 of Definition 48, so $r_m \upharpoonright \beta$ and $p_m \upharpoonright \beta$ would be incompatible, but this is a contradiction since both are in the generic filter). Since t is unique, it follows that $T_m \cap H$ has exactly two elements.

It will be convenient to introduce some notation. Given $K \subseteq T$, define $Last(K) = \{s(|s| - 1) \mid s \in K\}$. In other words, $Last(K)$ is obtained by collecting all the last component of every sequence in K.

As we saw above, if $s = \langle p_0, \ldots, p_m \rangle, t = \langle q_0, \ldots, q_m \rangle \in T_m \cap H$ and $s \neq t$, then $\triangle(s, t) = n$. Furthermore, by Lemma 49, we get that $p_i \upharpoonright \beta = q_i \upharpoonright \beta$ for every $i \leq m$. In this way, for every $m > n$ there must be $r_m \in \mathbb{P}_\beta$ and $\dot{x}_m, \dot{y}_m$ two $\mathbb{P}_\beta$-names such that $Last(T_m \cap H) = \{r_m {}^\frown \dot{x}_m, r_m {}^\frown \dot{y}_m\}$. Since T is an α-fusion tree, then $S = (\langle \dot{x}_m \rangle, \langle \dot{y}_m \rangle)_{m>n}$ is forced to be a 1-fusion sequence. By Lemma 20 there is $\dot{a}$ a $\mathbb{P}_\beta$-name for a condition in $\mathbb{S}(\mathcal{U}_\alpha)$ that is forced to be a lower bound of either $\langle \dot{x}_m \rangle_{m>n}$ or $\langle \dot{y}_m \rangle_{m>n}$ (note however, that it could happen that we do not know which possibility occurs without extending q). Clearly $q {}^\frown \dot{a}$ is the condition we were looking for.

Now, we consider the case where $\beta < \delta$. By the inductive hypothesis, there are $\dot{r}$ such that $q_1 = q {}^\frown \dot{r} \in \mathbb{P}_\delta$ and $\dot{R}'$ a $\mathbb{P}_\delta$-name for a branch through $T \upharpoonright \delta$ such that $q_1 \Vdash_\delta "\dot{R}' \subseteq \dot{G}_\delta"$ and $q_1 \Vdash_\delta "\dot{R}' \upharpoonright \beta = \dot{R}."$ Now, by the previous case, there are $\dot{r}_1$ such that $q_2 = q_1 {}^\frown \dot{r}_1 \in \mathbb{P}_\alpha$ and $\dot{R}''$ a $\mathbb{P}_\alpha$-name for a branch through T such that $q_2 \Vdash_\alpha "\dot{R}'' \subseteq \dot{G}_\alpha"$ and $q_2 \Vdash_\alpha "\dot{R}'' \upharpoonright \delta = \dot{R}'."$ This clearly finishes the proof of this case.

Assume α is a limit ordinal and let $\langle \delta_n \rangle_{n \in \omega} \subseteq M$ be an increasing sequence such that $\bigcup_{n \in \omega} \delta_n = \bigcup(M \cap \alpha)$, we also assume $\beta = \delta_0$ and $F_n \subseteq \delta_{n+1}$. We do the following:

1. By the inductive hypothesis, we can find $q_1 \in \mathbb{P}_{\delta_1}$ and $\dot{R}_1$ with the following properties:
   (a) $\dot{R}_1$ is a $\mathbb{P}_{\delta_1}$-name for a branch of the tree $T \upharpoonright \delta_1$.
   (b) $q_1 \upharpoonright \delta_0 = q_0$ (where $q_0 = q$).
   (c) $q_1 \Vdash "\dot{R}_1 \subseteq \dot{G}_{\delta_1}"$ and $q_1 \Vdash "\dot{R}_1 \upharpoonright \delta_0 = \dot{R}_0"$ (where $\dot{R}_0 = \dot{R}$).
2. By the inductive hypothesis, we can find $q_2 \in \mathbb{P}_{\delta_2}$ and $\dot{R}_2$ with the following properties:
   (a) $\dot{R}_2$ is a $\mathbb{P}_{\delta_2}$-name for a branch of the tree $T \upharpoonright \delta_2$.
   (b) $q_2 \upharpoonright \delta_1 = q_1$.
   (c) $q_2 \Vdash "\dot{R}_2 \subseteq \dot{G}_{\delta_2}"$ and $q_2 \Vdash "\dot{R}_2 \upharpoonright \delta_1 = \dot{R}_1."$

   $\vdots$ $\qquad\qquad\qquad\qquad$ $\vdots$

n+1. By the inductive hypothesis, we can find $q_{n+1} \in \mathbb{P}_{\delta_{n+1}}$ and $\dot{R}_{n+1}$ with the following properties:
   (a) $\dot{R}_{n+1}$ is a $\mathbb{P}_{\delta_{n+1}}$-name for a branch of the tree $T \upharpoonright \delta_{n+1}$.
   (b) $q_{n+1} \upharpoonright \delta_n = q_n$.
   (c) $q_{n+1} \Vdash "\dot{R}_{n+1} \subseteq \dot{G}_{\delta_n}"$ and $q_{n+1} \Vdash "\dot{R}_{n+1} \upharpoonright \delta_n = \dot{R}_n."$

For every $n \in \omega$, let $\dot{R}_n = \langle \dot{p}_i^n \rangle_{i \in \omega}$. By construction, we have that $\dot{p}_i^{n+1} \upharpoonright \delta_n$ and $\dot{p}_i^n$ are forced to be equal and $q_{n+1} \upharpoonright \delta_n = q_n$. Define $\overline{q} = \bigcup_{i \in \omega} q_i$ and $\dot{p}_i$ be a name for $\bigcup_{n \in \omega} \dot{p}_i^n$.

Let $\dot{R}' = \langle \dot{p}_i \rangle_{i \in \omega}$, we claim that $\overline{q}$ and $\dot{R}'$ have the desired properties. It is clear that $\overline{q} \upharpoonright \beta = q$ and $\overline{q} \Vdash "\dot{R}' \upharpoonright \beta = \dot{R}."$

We will now prove that $\dot{R}'$ is forced by $\overline{q}$ to be a branch through T. For every $m \in \omega$, let $\dot{R}'_m = \langle \dot{p}_0, \ldots, \dot{p}_m \rangle$. We need to prove that $\overline{q} \Vdash$“$\dot{R}'_m \in T$” for every $m \in \omega$. Assume this is not the case. In this way, we can find $m \in \omega$ such that $\overline{q} \Vdash$“$\dot{R}'_m \in T$” but $\overline{q} \not\Vdash$“$\dot{R}'_{m+1} \in T$” (recall point 1 in Definition 48). In this way, we can find $q_1 \leq \overline{q}$ such that $q_1 \Vdash$“$\dot{p}_{m+1} \notin suc_T(\dot{R}'_m)$.” Let $r \leq q_1$ and $s \in T$ such that $r \Vdash$“$s = \dot{R}'_m$.” Since $suc_T(s)$ is finite (has at most two elements in fact), we can find $r_1 \leq r$ and $\xi \in M \cap \alpha$ such that $r_1 \Vdash$“$\dot{p}_m \restriction \xi \neq x \restriction \xi$” for every $x \in suc_T(s)$. Pick $n \in \omega$ such that $\xi < \delta_n$. In this way, we get that $r_1 \restriction \delta_n$ forces that $\dot{R}'$ is not a branch trough $T \restriction \delta_n$, but this is a contradiction.

It remains to prove that $\overline{q} \Vdash$“$\dot{R}' \subseteq \dot{G}_\alpha$.” We need to prove that if $i \in \omega$, then $\overline{q} \Vdash$“$\dot{p}_i \in \dot{G}_\alpha$.” Recall that $\overline{q} = \bigcup_{i \in \omega} q_i$ and $\dot{p}_i$ be a name for $\bigcup_{n \in \omega} \dot{p}_i^n$. By construction, we have that $q_n \Vdash$“$\dot{p}_i^n \in \dot{G}_{\delta_n}$” for every $n \in \omega$, so we conclude that $\overline{q} \Vdash$“$\dot{p}_i \in \dot{G}_\alpha$.”   ∎

The next task is to be able to extend finite fusion trees. We will need a preliminary lemma.

Lemma 64   *Let $p \in \mathbb{P}_{\beta+1}, F \in [\beta+1]^{<\omega}, \eta : F \longrightarrow \omega$ and $D \subseteq \mathbb{P}_{\beta+1}$ be $(F, \eta)^*$-open dense below p. The set*

$$L = \{ r \in \mathbb{P}_\beta \mid \exists \dot{x} \, (r \hat{\ } \dot{x} \leq^*_{F, \eta} p \wedge r \hat{\ } \dot{x} \in D) \}$$

is $(F', \eta')^*$-open dense below $p \restriction \beta$ (where $F' = F \cap \beta$ and $\eta' = \eta \restriction F$).

Proof   We first show that L is $(F', \eta')^*$-open below $p \restriction \beta$. Let $q \in L$ and $r \leq^*_{F', \eta'} q$. Since $q \in L$, there is $\dot{x}$ such that $q \hat{\ } \dot{x} \leq^*_{F, \eta} p$ and $q \hat{\ } \dot{x} \in D$. It follows that $r \hat{\ } \dot{x} \leq^*_{F, \eta} q \hat{\ } \dot{x}$, so $r \hat{\ } \dot{x} \leq^*_{F, \eta} p$. Since D is $(F, \eta)^*$-open and $q \hat{\ } \dot{x} \in D$, we get that $r \hat{\ } \dot{x} \in D$ which implies that $r \in L$.

We will now prove that L is $(F', \eta')^*$-dense below $p \restriction \beta$. Let $q \leq^*_{F', \eta'} p \restriction \beta$, define $q_1 = q \hat{\ } p(\beta)$ and note that $q_1 \leq^*_{F, \eta} p$. Since D is $(F, \eta)^*$-dense, there is $q_2 \leq^*_{F, \eta} q_1$ such that $q_2 \in D$. Let $r = q_2 \restriction \beta$. It follows that $r \in L$ since $q_2 = r \hat{\ } q_2(\beta)$.   ∎

We conclude with the following key result, which will allow us to keep extending finite trees.

Lemma 65   *Let $\alpha \in M \preceq \mathsf{H}(\kappa), L = \{ (F_n, \eta_n) \mid n \in \omega \}$ suitable for (M, α) and T an α-finite fusion tree of height n. For every $s = \langle p_0, \ldots, p_n \rangle \in T$ let $D_s \in M$ be an $(F_n, \eta_n)^*$-open dense set below p_n. Then there is $\widetilde{T}$ with the following properties:*

1. *$\widetilde{T}$ is a α-finite fusion tree of height $n + 1$.*
2. *$\widetilde{T}$ is an end-extention of T.*
3. *If $s = \langle p_0, \ldots, p_n \rangle \in T$, then $suc_{\widetilde{T}}(s) \subseteq D_s$.*

Proof   We prove it by induction on α. The case $\alpha = 1$ follows by the argument of Lemma 17. Now assume $\alpha = \beta + 1$ and the lemma holds for β, let $S = T \restriction \beta$. For every $m \in \omega$, let $F'_m = F_m \cap \beta$ and $\eta'_m = \eta_m \restriction F'_m$. For the proof in this case, given $s = \langle p_0, \ldots, p_n \rangle \in T$, we will denote $s' = \langle p_0 \restriction \beta, \ldots, p_n \restriction \beta \rangle$. There are several cases to consider:

Case 66 $\beta \notin F_{n+1}$.

In this case, we have that $F'_{n+1} = F_{n+1}$ and $F'_n = F_n$. This is the easiest case since "β has not come into play yet." We provide the details.

Let $s = \langle p_0^s, \dots, p_n^s \rangle \in T$. Define

$$L_s = \left\{ q \in \mathbb{P}_\beta \mid \exists \dot{x} \left(q^\frown \dot{x} \leq^*_{F_n, \eta_n} p_n^s \wedge q^\frown \dot{x} \in D_s \right) \right\}.$$

By Lemma 64, we know that L_s is $(F_n, \eta_n)^*$-open dense below $p_n^s \upharpoonright \beta$.

We now claim that if $s = \langle p_0, \dots, p_n \rangle, t = \langle q_0, \dots, q_n \rangle \in T$ and $s \neq t$, then $s' = \langle p_0 \upharpoonright \beta, \dots, p_n \upharpoonright \beta \rangle$ and $t' = \langle q_0 \upharpoonright \beta, \dots, q_n \upharpoonright \beta \rangle$ are different. Let $m = \triangle(s, t)$, we know that $p_m \perp q_m$ and since $\beta \notin F_{m+1}$, it follows that $p_m \upharpoonright \beta \perp q_m \upharpoonright \beta$ (and in particular, they are different).

By the remark above, for every $z = \langle p'_0, \dots, p'_n \rangle \in S$, there is an unique $s = \langle p_0, \dots, p_n \rangle \in T$ such that $s' = z$. We also know that L_s is $(F_n, \eta_n)^*$-open dense below p'_n. By the inductive hypothesis, there is a fusion tree $\widehat{T}$ such that the following holds:

1. $\widehat{T}$ is a β-finite fusion tree of height $n + 1$.
2. $\widehat{T}$ is an end-extention of S.
3. If $s \in T_n$, then $suc_{\widehat{T}}(s') \subseteq L_s$.

We now define the tree $\widetilde{T}$ as follows:

1. $\widetilde{T}$ is a tree of height $n + 1$.
2. $\widetilde{T}$ is an end-extention of T.
   Let $s = \langle p_0, \dots, p_n \rangle \in T$.
3. We know that $d_{s'}^{\widehat{T}}$ and $w_{s'}^{\widehat{T}}$ are elements of L_s. This means that there are $\dot{x}_s$ and $\dot{y}_s$ such that $d_{s'}^{\widehat{T}} {}^\frown \dot{x}_s$ and $w_{s'}^{\widehat{T}} {}^\frown \dot{y}_s$ are $\leq^*_{F_n, \eta_n}$-extensions of p_n and are in D_s. Define $d_s^T = d_{s'}^{\widehat{T}} {}^\frown \dot{x}_s$ and $w_s^T = w_{s'}^{\widehat{T}} {}^\frown \dot{y}_s$ (in case that $F_n = F_{n+1}$ we have that $F'_n = F'_{n+1}$. The definition of fusion tree demands that $d_{s'}^{\widehat{T}} = w_{s'}^{\widehat{T}}$. We take $\dot{x}_s = \dot{y}_s$ so that we get $d_s^T = w_s^T$).

We need to prove that $\widetilde{T}$ is an α-finite fusion tree, but this is easy since $\widehat{T}$ is a β-finite fusion tree and $\beta \notin F_{n+1}$. This finishes the proof for this case.

Case 67 $\beta \in F_{n+1} \backslash F_n$.

In this case, we have that $F'_{n+1} = F_n$ and $F'_n = F_n$.

As in the previous case, for every $s = \langle p_0^s, \dots, p_n^s \rangle \in T$, define $L_s = \{ q \in \mathbb{P}_\beta \mid \exists \dot{x} \left(q^\frown \dot{x} \leq^*_{F_n, \eta_n} p_n^s \wedge q^\frown \dot{x} \in D_s \right) \}$. Again, by Lemma 64, we already know that this set is $(F'_n, \eta'_n)^*$-open dense below $p_n^s \upharpoonright \beta$. Now, note that if $q \in L_s$ and $\dot{x}$ is such that $q^\frown \dot{x} \leq^*_{F_n, \eta_n} p_n^s$ and $q^\frown \dot{x} \in D_s$, we can now take $\dot{a}$ and $\dot{b}$ be two $\mathbb{P}_\beta$-names for conditions on $\mathbb{S}(\dot{\mathcal{U}}_\beta)$ such that q forces that they are incompatible extensions of $\dot{x}$, yet $E_{\dot{a}}$ and $E_{\dot{b}}$.

In this way, if $q \in L_s$, we can find $\dot{a}$ and $\dot{b}$ with the following properties:

1. $q^\frown \dot{a} \leq^*_{F_n, \eta_n} p_n^s$ and $q^\frown \dot{b} \leq^*_{F_n, \eta_n} p_n^s$.
2. $q^\frown \dot{a}, q^\frown \dot{b} \in D_s$.
3. $q \Vdash \text{``} \dot{a} \perp \dot{b} \wedge E_{\dot{a}} = E_{\dot{b}} \text{.''}$

The second point follows because D_s is $(F_n, \eta_n)^*$-open below p_n^s.

In the same way as in the previous case, we have that if $s = \langle p_0^s, \dots, p_n^s \rangle$, $t = \langle p_0^t, \dots, p_n^t \rangle \in T$ and $s \neq t$, then $s' \neq t'$. Once again, for every $z \in S_n$, there is a

unique $s \in T_n$ such that $s' = z$. We also know that L_s is $(F_n, \eta_n)^*$-open dense below $p_n^s \upharpoonright \beta$. By the inductive hypothesis, there is a tree $\widehat{T}$ such that the following holds:

1. $\widehat{T}$ is a β-finite fusion tree of height $n + 1$.
2. $\widehat{T}$ is an end-extension of S.
3. If $s = \langle p_0^s, \ldots, p_n^s \rangle \in T_n$, then $d_{s'}^{\widehat{T}} = w_{s'}^{\widehat{T}}$ and it belongs to L_s (recall that $F_n' = F_{n+1}' = F_n$ holds in this case).

Let $s = \langle p_0^s, \ldots, p_n^s \rangle \in T$, we know that $s' \in \widehat{T}$ and $suc_{\widehat{T}}(s')$ has only one element. Since $d_{s'}^{\widehat{T}} \in \overline{L}_s$, by the above remark, we know that there are $\dot{a}_s$ and $\dot{b}_s$ such that the following holds:

1. $d_{s'}^{\widehat{T}} {}^\frown \dot{a}_s \leq_{F_n, \eta_n}^* p_n^s$ and $d_{s'}^{\widehat{T}} {}^\frown \dot{b}_s \leq_{F_n, \eta_n}^* p_n^s$.
2. $d_{s'}^{\widehat{T}} {}^\frown \dot{a}_s, d_{s'}^{\widehat{T}} {}^\frown \dot{b}_s \in D_s$.
3. $d_{s'}^{\widehat{T}} \Vdash \text{``} \dot{a}_s \perp \dot{b}_s \wedge E_{\dot{a}_s} = E_{\dot{b}_s}. \text{''}$

Recall that S_n is an antichain (see Lemma 49). In this way, using Lemma 45, we can find two $\mathbb{P}_\beta$-names $\dot{e}_n^T$ and $\dot{u}_n^T$ such that if $s = \langle p_0^s, \ldots, p_n^s \rangle \in T$, then $p_n^s \upharpoonright \beta \Vdash \text{``} \dot{e}_n^T = \dot{a}_s \wedge \dot{u}_n^T = \dot{b}_s. \text{''}$ Define $d_s^T = d_{s'}^{\widehat{T}} {}^\frown \dot{e}_n^T$ and $w_s^T = d_{s'}^{\widehat{T}} {}^\frown \dot{u}_n^T$ (which is the same as $w_{s'}^{\widehat{T}} {}^\frown \dot{u}_n^T$). Define the tree $\widetilde{T}$ as follows:

1. $\widetilde{T}$ is a tree of height $n + 1$.
2. $\widetilde{T}$ is an end-extension of T.
3. If $s = \langle p_0, \ldots, p_n \rangle \in T$, then $suc_{\widetilde{T}}(s) = \{d_s^T, w_s^T\}$.

It is easy to see that $\widetilde{T}$ has the desired properties. This finishes the proof for this case.

Case 68 $\beta \in F_n$.

In this case, we have that $F_{n+1}' = F_{n+1} \setminus \{\beta\}$ and $F_n' = F_n \setminus \{\beta\}$.

Let $s = \langle p_0^s, \ldots, p_n^s \rangle \in T$ and note that there is a unique $t = \langle p_0^t, \ldots, p_n^t \rangle \in T$ such that $s' = t'$ and $s \neq t$ (see Lemma 60). Let $z = s' = t'$ (which is a node in S). Define L_z as the set of all $q \in \mathbb{P}_\beta$ such that there are $\dot{x}, \dot{y}$ with the following properties:

1. $q \leq_{F_n', \eta_n'}^* p_n^s \upharpoonright \beta$ (recall that $p_n^s \upharpoonright \beta = p_n^t \upharpoonright \beta$).
2. $q {}^\frown \dot{x} \leq_{F_n, \eta_n}^* p_n^s$ and $q {}^\frown \dot{y} \leq_{F_n, \eta_n}^* p_n^t$.
3. $q {}^\frown \dot{x} \in D_s$ and $q {}^\frown \dot{y} \in D_t$.
4. $q \Vdash \text{``} \langle p_n^s(\beta), p_n^t(\beta), \dot{x}, \dot{y} \rangle$ is an $\eta_n(\beta)$-nice sequence''.

We claim that L_z is $(F_n', \eta_n')^*$-open dense below $p_n^s \upharpoonright \beta$. It is easy to see that L_z is $(F_n', \eta_n')^*$-open below $p_n^s \upharpoonright \beta$. The proof that it is also dense is more elaborate, we basically follow the argument of Lemma 17. It will be convenient for the reader to review the proof of this lemma before continuing.

Let $r \leq_{F_n', \eta_n'}^* p_n^s \upharpoonright \beta$, we need to find an $\leq_{F_n', \eta_n'}^*$-extension that is in L_z. Since $r {}^\frown p_n^s(\beta)$ is an $\leq_{F_n, \eta_n}^*$-extension of p_n^s and D_s is $\leq_{F_n, \eta_n}^*$-dense below it, we can find $r_1 \in \mathbb{P}_\beta$ and $\dot{x}$ a $\mathbb{P}_\beta$-name for an element of $\mathbb{S}(\dot{\mathcal{U}}_\alpha)$ such that $r_1 {}^\frown \dot{x} \leq_{F_n, \eta_n}^* p_n^s$ and $r_1 {}^\frown \dot{x} \in D_s$. Let $\dot{U}$ be the name for the following set:

$$\left(dom\left(p_n^s(\beta)\right) \setminus dom\left(\dot{x}\right) \right) \cup \left([\dot{a}_{\eta_n(\beta)}]_{\dot{x}} \setminus [\dot{a}_{\eta_n(\beta)}]_{p_n^s(\beta)} \right).$$

Now, let $\dot{y}$ be the $\mathbb{P}_\beta$-name for the extension of $p_n^t(\beta)$ obtained by adding $\dot{U}$ to the $\eta_n(\beta)$-class of the partition of $p_n^t(\beta)$. Note that $r_1 {}^\frown \dot{y} \leq_{F_n, \eta_n}^* p_n^t$. Using the density of

D_t, we can now find $\dot{y}_1$ and q such that $q^\frown \dot{y}_1 \leq^*_{F_n, \eta_n} r_1^\frown \dot{y}$ and $q^\frown \dot{y}_1 \in D_t$. Let $\dot{U}_1$ be the name for the following set:

$$(dom(\dot{y}) \backslash dom(\dot{y}_1)) \cup \left([\dot{a}_{\eta_n(\beta)}]_{\dot{y}_1} \backslash [\dot{a}_{\eta_n(\beta)}]_{\dot{y}} \right).$$

Now, let $\dot{x}_1$ be the $\mathbb{P}_\beta$-name for the extension of $\dot{x}$ obtained by adding $\dot{U}_1$ to the $\eta_n(\beta)$-class of the partition of $\dot{x}$. It follows that q has the desired properties (note that $q^\frown \dot{x}_2 \in D_s$ since $q^\frown \dot{x}_2 \leq^*_{F_n, \eta_n} r_1^\frown \dot{x}$).

For every $z \in S_n$ we have the set L_z. By the inductive hypothesis, there is a tree $\widehat{T}$ such that the following holds:

1. $\widehat{T}$ is a β-finite fusion tree of height $n + 1$.
2. $\widehat{T}$ is an end-extention of T'.
3. If $z \in \widehat{T}_n$, then $suc_{\widehat{T}}(z) \subseteq L_z$.

We now define the tree $\widetilde{T}$ as follows:

1. $\widetilde{T}$ is a tree of height $n + 1$.
2. $\widetilde{T}$ is an end-extension of T.
   Let $s = \langle p_0^s, \dots, p_n^s \rangle, t = \langle p_0^t, \dots, p_n^t \rangle \in T$ with $z = s' = t'$.
3. In case $F_n = F_{n+1}$, we have that $F'_n = F'_{n+1}$. Since $d_z^{\widehat{T}} \in suc_{\widehat{T}}(z) \subseteq L_z$, we know that there are $\dot{x}$ and $\dot{y}$ with the following properties:
   (a) $d_z^{\widehat{T}} \leq^*_{F'_n, \eta'_n} p_n^s \upharpoonright \beta$.
   (b) $d_z^{\widehat{T}} {}^\frown \dot{x} \leq^*_{F_n, \eta_n} p_n^s$ and $d_z^{\widehat{T}} {}^\frown \dot{y} \leq^*_{F_n, \eta_n} p_n^t$.
   (c) $d_z^{\widehat{T}} {}^\frown \dot{x} \in D_s$ and $d_z^{\widehat{T}} {}^\frown \dot{y} \in D_t$.
   (d) $d_z^{\widehat{T}} \Vdash$ "$\langle p_n^s(\beta), p_n^t(\beta), \dot{x}, \dot{y} \rangle$ is an $\eta_n(\beta)$-nice sequence."
       Define $d_s^T = w_s^T = d_z^{\widehat{T}} {}^\frown \dot{x}$ and $d_t^T = w_t^T = d_z^{\widehat{T}} {}^\frown \dot{y}$.
4. In case $F_n \neq F_{n+1}$, first note that $F'_n \neq F'_{n+1}$ as well (if $\gamma \in F_{n+1} \backslash F_n$, then $\gamma \neq \beta$ so $\gamma \in F'_{n+1} \backslash F'_n$). Since $suc_{\widehat{T}}(z) \subseteq L_z$, we know there are $\dot{x}_0, \dot{y}_0, \dot{x}_1$, and $\dot{y}_1$ such that the following holds:
   (a) $d_z^{\widehat{T}}, w_z^{\widehat{T}} \leq^*_{F'_n, \eta'_n} p_n^s \upharpoonright \beta$.
   (b) $d_z^{\widehat{T}} {}^\frown \dot{x}_0, w_z^{\widehat{T}} {}^\frown \dot{x}_1 \leq^*_{F_n, \eta_n} p_n^s$ and $d_z^{\widehat{T}} {}^\frown \dot{y}_0, w_z^{\widehat{T}} {}^\frown \dot{y}_1 \leq^*_{F_n, \eta_n} p_n^t$.
   (c) $d_z^{\widehat{T}} {}^\frown \dot{x}_0 \in D_s, d_z^{\widehat{T}} {}^\frown \dot{y}_0 \in D_t$ and $w_z^{\widehat{T}} {}^\frown \dot{x}_1 \in D_s, w_z^{\widehat{T}} {}^\frown \dot{y}_1 \in D_t$.
   (d) $d_z^{\widehat{T}} \Vdash$ "$\langle p_n^s(\beta), p_n^t(\beta), \dot{x}_0, \dot{y}_0 \rangle$ is an $\eta_n(\beta)$-nice sequence" and $w_z^{\widehat{T}} \Vdash$ "$\langle p_n^s(\beta), p_n^t(\beta), \dot{x}_1, \dot{y}_1 \rangle$ is an $\eta_n(\beta)$-nice sequence."
       We now define $d_s^T = d_z^{\widehat{T}} {}^\frown \dot{x}_0, w_s^T = w_z^{\widehat{T}} {}^\frown \dot{x}_1$ and $d_t^T = d_z^{\widehat{T}} {}^\frown \dot{y}_0, w_t^T = w_z^{\widehat{T}} {}^\frown \dot{y}_1$.

We need to prove that $\widetilde{T}$ is an α-finite fusion tree. Points 1–3 of Definition 48 hold by construction, while point 4 holds since $\widehat{T}$ is a β-fusion tree. We now prove that point 5 also holds. Let $s = \langle p_0^s, \dots, p_n^s \rangle, t = \langle p_0^t, \dots, p_n^t \rangle \in T, m = \triangle(s, t), \gamma \in F_{m+1} \backslash F_m$ and assume that $p_n^s \upharpoonright \gamma = p_n^t \upharpoonright \gamma$ holds. In case $\gamma = \beta$ (so $m = n$), we get the desired conclusion by construction, if $\gamma < \beta$, we get the conclusion because $\widehat{T}$ is a β-fusion tree. This finishes the proof for this case.

We conclude that the result is true for successor ordinals.

Finally, assume α is a limit ordinal and the conclusion holds for every ordinal less than α. Choose $\beta \in M \cap \alpha$ such that $F_{n+1} \subseteq \beta$. For the rest of the proof, if $t = \langle q_0, \ldots, q_n \rangle \in T$, we will denote $t' = \langle q_0 \restriction \beta, \ldots, q_n \restriction \beta \rangle$. By the same arguments as before, it is easy to prove that if $s = \langle p_0, \ldots, p_n \rangle, t = \langle q_0, \ldots, q_n \rangle \in T$ and $s \neq t$, then $s' \neq t'$.

For every $s = \langle p_0, \ldots, p_n \rangle \in T$, define the set

$$L_s = \left\{ q \in \mathbb{P}_\beta \mid \exists \dot{x} \left(q^\frown \dot{x} \leq^*_{F_n, \eta_n} p_n \wedge q^\frown \dot{x} \in D_s \right) \right\}.$$

With the same argument as Lemma 64, we can prove that L_s is $(F_n, \eta_n)^*$-open and $(F_n, \eta_n)^*$-dense below $p_n \restriction \beta$. Let $S = T \restriction \beta$.

For every $s' \in S_n$, we have L_s. By the inductive hypothesis, there is a tree $\widehat{T}$ such that the following holds:

1. $\widehat{T}$ is a β-finite fusion tree of height $n + 1$.
2. $\widehat{T}$ is an end-extension of S.
3. If $s' \in \widehat{T}_n$, then $suc_{\widehat{T}}(s') \subseteq L_s$.

We now define the tree $\widetilde{T}$ as follows:

1. $\widetilde{T}$ is a tree of height $n + 1$.
2. $\widetilde{T}$ is an end-extension of T.
        Let $s = \langle p_0, \ldots, p_n \rangle \in T$.
3. Case $F_n = F_{n+1}$. Since $d_s^{\widehat{T}} \in L_s$, there is $\dot{x}$ such that $d_s^{\widehat{T}} {}^\frown \dot{x} \leq^*_{F_n, \eta_n} p_n$ and $d_s^{\widehat{T}} {}^\frown \dot{x} \in D_s$. Define $d_s^T = d_s^{\widehat{T}} {}^\frown \dot{x}$.
4. Case $F_n \neq F_{n+1}$. We know there are $\dot{x}$ and $\dot{y}$ such that $d_s^{\widehat{T}} {}^\frown \dot{x} \leq^*_{F_n, \eta_n} p_n$ with $d_s^{\widehat{T}} {}^\frown \dot{x} \in D_s$ and also $w_s^{\widehat{T}} {}^\frown \dot{y} \leq^*_{F_n, \eta_n} p_n$ with $w_s^{\widehat{T}} {}^\frown \dot{y} \in D_s$. Define $d_s^T = d_s^{\widehat{T}} {}^\frown \dot{x}$ and $w_s^T = w_s^{\widehat{T}} {}^\frown \dot{y}$.

We need to prove that $\widetilde{T}$ is a α-finite fusion tree, but this is easy since $\widehat{T}$ is a β-finite fusion tree and $F_{n+1} \subseteq \beta$ (so everything interesting happens before β). This finishes the proof for this case and the proof overall. ∎

5 Preserving $c_{\min}$-covering at successor steps

With the tools developed in the last section, we can finally start the proof that the iteration of the Shelah's forcing preserves $c_{\min}$-covering. The proof will be by induction. The base case has already been done, we will do the successor step in this section and the limit step in the next one. This proof takes inspiration in the result by Geschke that the iteration of the Miller lite forcing preserves $c_{\min}$-covering (see [7]). For this section, assume $\alpha = \beta + 1$ and $\dot{x}$ is the name of a real that was not added at the β-step.

Definition 69 Let T and S be two finite subtrees of $2^{<\omega}$ and $i < 2$. We say that $c_{\min}(T, S) = i$ if for every maximal node t of T and every maximal node s of S the following holds:

1. t and s are incompatible.
2. $c_{\min}(t, s) = i$.

Note that if $\mathsf{c}_{\min}(T, S) = i$ and $\overline{T}$ and $\overline{S}$ are end-extentions of T and S respectively, then $\mathsf{c}_{\min}(\overline{T}, \overline{S}) = i$ holds as well. Recall that $\mathbb{P}$ is a partial order, $\dot{y}$ is a $\mathbb{P}$-name for an element in 2^ω and $p \in \mathbb{P}$, we defined $\dot{y}[p] = \bigcup\{t \mid p \Vdash \text{``}t \subseteq \dot{y}''\}$. Also remember that if $p \in \mathbb{S}(\mathcal{U})$, we defined $S_p(\dot{y}) = \bigcup\{\dot{y}[p[h]] \mid h \in 2^{<\omega}\}$.

Definition 70 Let $q \in \mathbb{P}_\beta$ and $\dot{p}$ a $\mathbb{P}_\beta$-name for a condition of $\mathbb{S}(\dot{\mathcal{U}}_\beta)$ and $n \in \omega$. We say that q *determines* $\dot{p}$ *to the n-level* if there is a tree $T_n^{\dot{p}}(q)$ such that $q \Vdash \text{``}T_n^{\dot{p}}(q) = \bigcup\{\dot{x}[p[h]] \mid h \in 2^{n+1}\}$.''

The name above may be a little misleading, since q is not really determining p, but the tree of possibilities of $\dot{x}$ to the n-branching level. However, we prefer this name instead of something more descriptive, yet much more notationally complicated (like "q determines $\dot{S}_p(x) \restriction Split_n$" or something similar).

By the results of the previous sections, we know that if $q \in \mathbb{P}_\beta$ and $\dot{p}_1$ is a $\mathbb{P}_\beta$-name for a condition of $\mathbb{S}(\dot{\mathcal{U}}_\beta)$, then we can find $\dot{p}_2$ a $\mathbb{P}_\beta$-name for a condition of $\mathbb{S}(\dot{\mathcal{U}}_\beta)$ such that q forces that $\dot{p}_2$ is an extension of $\dot{p}_1$, $\dot{p}_2$ is ω-separative and $S_{\dot{p}_2}(\dot{x})$ is forced to be a monochromatic Sacks tree (although we might not know of which color without extending q first).

Definition 71 Let $q \in \mathbb{P}_\beta$ and $\dot{p}$ a $\mathbb{P}_\beta$-name for a condition of $\mathbb{S}(\dot{\mathcal{U}}_\beta)$. Let $F \in [\beta]^{<\omega}$ and $\eta : F \longrightarrow \omega$. We say $(q, \dot{p})$ is (F, η, n, i)-*faithful* if the following holds:

1. q forces that $\dot{p}$ is ω-separative and $S_{\dot{p}}(\dot{x})$ is an i-monochromatic Sacks tree.
2. If $\sigma \in \prod_{\gamma \in F} 2^{\eta(\gamma)}$ then $q * \sigma$ determines $\dot{p}$ to the n-level.
3. If $\sigma, \sigma' \in \prod_{\gamma \in F} 2^{\eta(\gamma)}$ and $\sigma \neq \sigma'$ then $\mathsf{c}_{\min}(T_n^{\dot{p}}(q * \sigma), T_n^{\dot{p}}(q * \sigma')) = i$ (where $T_n^{\dot{p}}(q * \sigma)$ and $T_n^{\dot{p}}(q * \sigma')$ are as in Definition 70).

It is important to remark the following:

Lemma 72 Let $F \in [\beta]^{<\omega}$, $\eta : F \longrightarrow \omega$ and $(q, \dot{p})$ be (F, η, n, i)-faithful. If $r \leq_{F,\eta}^* q$, then $(r, \dot{p})$ is (F, η, n, i)-faithful.

We will now prove the next lemma:

Lemma 73 Let $(q, \dot{p})$ be $(F, \eta, n+1, i)$-faithful. There are $\dot{p}^0, \dot{p}^1$ with the following properties:

1. $(q, \dot{p}^0)$ and $(q, \dot{p}^1)$ are (F, η, n, i)-faithful.
2. $q \Vdash \text{``}\dot{p}^0, \dot{p}^1 \leq_n^{**} \dot{p}$.''
3. $\mathsf{c}_{\min}(T_n^{\dot{p}^0}(q * \sigma), T_n^{\dot{p}^1}(q * \sigma)) = i$ whenever $\sigma \in \prod_{\gamma \in F} 2^{\eta(\gamma)}$.

Proof Assume $(q, \dot{p})$ is $(F, \eta, n+1, i)$-faithful. Let $\dot{p}^0$ be (the name of) the condition obtained by merging $\left[a_{\dot{p}}(n)\right]_p$ with $\left[a_{\dot{p}}(n+1)\right]_p$ and $a_{\dot{p}}(n+1) \in Lov_{p(\beta)}(a_{\dot{p}}(n))$ and let $\dot{p}^1$ be (the name of) the condition obtained by merging $\left[a_{\dot{p}}(n)\right]_p$ with $\left[a_{\dot{p}}(n+1)\right]_p$ and $a_{\dot{p}}(n+1) \in Hat_{p(\beta)}(a_{\dot{p}}(n))$. It follows that $(q, \dot{p}^0)$ and $(q, \dot{p}^1)$ are (F, η, n, i)-faithful. It is also easy to see that if $\sigma \in \prod_{\gamma \in F} 2^{\eta(\gamma)}$, then

$$\mathsf{c}_{\min}(T_n^{\dot{p}^0}(q * \sigma), T_n^{\dot{p}^1}(q * \sigma)) = i. \qquad \blacksquare$$

We will need the following:

Lemma 74 *Let* $(q, \dot{p})$ *be* (F, η, n, i)-*faithful.*

1. *The set* $D = \{ r \leq_{F, \eta}^* q \mid (r, \dot{p}) \text{ is } (F, \eta, n + 1, i)\text{-faithful}\}$ *is* $(F, \eta)^*$-*open dense below* q.
2. *For every* $m \in \omega$ *with* $m > n$, *the set* $D_m = \{ r \leq_{F, \eta}^* q \mid (r, \dot{p}) \text{ is } (F, \eta, m, i)\text{-faithful}\}$ *is* $(F, \eta)^*$-*open dense below* q.

Proof Obviously the first point implies the second one. It is easy to see that D is $(F, \eta)^*$-open. We will now prove that D is $(F, \eta)^*$-dense. Let $q_1 \leq_{F, \eta}^* q$ and fix an enumeration $\prod\limits_{\gamma \in F} 2^{\eta(\gamma)} = \{ \sigma_j \mid j < k \}$. We can recursively build conditions $q_1^0, \dots, q_1^{k+1}$ with the following properties:

1. $\langle q_1^0, \dots, q_1^{k+1} \rangle$ is an $\leq_{(F, \eta)}$-decreasing sequence.
2. $q_1^0 = q_1$.
3. $q_1^{i+1} * \sigma_j$ determines $\bigcup \{ \dot{x} (p[h]) \mid h \in 2^{n+2} \}$.

We claim that $r = q_1^{k+1}$ is the condition we are looking for. In order to prove this, note that if $j_1 \neq j_2$, then $\mathsf{c}_{\min}(T_{n+1}^{\dot{p}} (q_1^{k+1} * \sigma_{j_1}), T_{n+1}^{\dot{p}} (q_1^{k+1} * \sigma_{j_2})) = i$ because $T_{n+1}^{\dot{p}} (q_1^{k+1} * \sigma_{j_1})$ and $T_{n+1}^{\dot{p}} (q_1^{k+1} * \sigma_{j_2})$ are end-extentions of trees that already have that property. ∎

The following lemma is easy and follows from the definitions:

Lemma 75 *Let* $(q, \dot{p})$ *be* (F, η, n, i)-*faithful and* $\gamma \notin F$. *Define* $F' = F \cup \{ \gamma \}$ *and* $\eta' = \eta \cup \{ (\gamma, 0) \}$. *The condition* $(q, \dot{p})$ *is* (F', η', n, i)-*faithful.*

We will need one more lemma:

Lemma 76 *Assume* $(q, \dot{p})$ *is* $(F, \eta, n + 1, i)$-*faithful. Let* $\delta \in F$ *and define* $\eta' : F \longrightarrow \omega$ *where* $\eta'(\xi) = \eta(\xi)$ *if* $\xi \neq \delta$ *and* $\eta'(\delta) = \eta(\delta) + 1$. *There is* $\dot{p}_\#$ *such that* $q \Vdash$ "$\dot{p}_\# \leq_n^{**} \dot{p}$" *and* $(q, \dot{p}_\#)$ *is* (F, η', n, i)-*faithful.*

Proof Let $\dot{p}^0$ and $\dot{p}^1$ be as in Lemma 73. Now, define a $\mathbb{S}(\dot{\mathcal{U}}_\beta)$-name $\dot{p}_\#$ such that if $\sigma \in \prod\limits_{\gamma \in F} 2^{\eta'(\gamma)}$ then $q * \sigma \Vdash$ "$\dot{p}_\# = \dot{p}^{\sigma(\delta)(\eta(\gamma))}$" (In other words, if the last digit of $\sigma(\delta)$ is k, then $q * \sigma \Vdash$"$\dot{p}_\# = \dot{p}^k$"). It follows that $(q, \dot{p}_\#)$ is (F, η', n, i)-faithful. ∎

We can now prove the most important result on this section:

Proposition 77 *Let* $\alpha = \beta + 1$ *and assume* $\dot{x}$ *is a* $\mathbb{P}_\alpha$-*name for a real that was not added by* $\mathbb{P}_\beta$. *If* $q' \in \mathbb{P}_\alpha$, *then there are* $\overline{q} \leq q'$ *and* $S \in V$ *a* $\mathsf{c}_{\min}$-*monochromatic tree such that* $\overline{q} \Vdash$ "$\dot{x} \in [S]$."

Proof We may assume that there are $q \in \mathbb{P}_\beta$ and $\dot{p}$ a $\mathbb{P}_\beta$-name for a condition of $\mathbb{S}(\dot{\mathcal{U}}_\beta)$ such that q forces that $\dot{p}$ is ω-separative, $S_{\dot{p}}(\dot{x})$ is an i-monochromatic Sacks tree (for some $i < 2$) and $q' = q^\frown \dot{p}$. Let M be a countable elementary submodel of $\mathsf{H}(\kappa)$ (for a big large enough regular cardinal κ) such that $\alpha, q, \dot{p}, \dot{\mathcal{U}}_\beta, \dot{x} \in M$.

For every $l < 3$, define $A_l = \{n \mid n =_3 l\}$ (where $n =_3 l$ means that n is equal to l mod 3). We now find $L = \{(F_n, \eta_n) \mid n \in \omega\}$ suitable for (M, β) with the following extra properties:

1. $F_0 = \emptyset$ and $\eta_0 = \emptyset$.
2. Let $n \in \omega$ and assume F_n, η_n are already defined, we will now define F_{n+1} and η_{n+1}.
   (a) **Case $n \in A_0$.**
       Let $F_{n+1} = F_n$ and $\eta_{n+1} = \eta_n$.
   (b) **Case $n \in A_1$.**
       Let $F_{n+1} = F_n$ and there is a unique $\xi \in F_n$ such $\eta_{n+1}(\xi) = \eta_n(\xi) + 1$ and $\eta_{n+1}(\gamma) = \eta_n(\gamma)$ for every other $\gamma \in F_n$.
   (c) **Case $n \in A_2$.**
       $F_{n+1} \backslash F_n$ has size at most $1, \eta_{n+1} \upharpoonright F_n = \eta_n \upharpoonright F_n$ and if $\gamma \in F_{n+1} \backslash F_n$, then $\eta_{n+1}(\gamma) = 0$.

Define a tree $Y \subseteq 2^{<\omega}$ as follows:

1. $\emptyset \in Y$.
2. If $s \in Y_n$ and $F_{n+1} = F_n$, then $s^\frown 0 \in Y_{n+1}$.
3. If $s \in Y_n$ and $F_{n+1} \neq F_n$, then $s^\frown 0, s^\frown 1 \in Y_{n+1}$.

It follows that if $s \in Y$ is a splitting node, then $|s| \in A_2$ (but not every node whose height is in A_2 must be a splitting node).

We will now recursively build a β-fusion tree $T = \{q_s \mid s \in Y\} \subseteq M$, $\{k_n \mid n \in \omega\} \subseteq \omega$ and $\{\dot{p}_s \mid s \in Y\} \subseteq M$ with the following properties[6]:

1. $k_0 = 0$.
2. $q_\emptyset \leq q$ and $q_\emptyset \Vdash "\dot{p}_\emptyset \leq \dot{p}."$
3. if $s \in Y_n$, then $(q_s, \dot{p}_s)$ is (F_n, η_n, k_n, i)-faithful. Define $K(q_s) = \bigcup \{T_n^{\dot{p}_s}(q_s * \sigma) \mid \sigma \in \prod_{\gamma \in F_n} 2^{\eta_n(\gamma)}\}$.
4. If $s \in Y_n$ and $j \in suc_Y(s)$, then $q_{s^\frown j} \leq^*_{F_n, \eta_n} q_s$ and $q_{s^\frown j} \Vdash "\dot{p}_{s^\frown j} \leq^{**}_{k_n} \dot{p}''_s$.
5. If $s \in Y$ and $j \in suc_Y(s)$, then $K(q_{s^\frown j})$ is an end-extension of $K(q_s)$.
6. If $s \in Y$ and $s^\frown 0, s^\frown 1 \in Y$, then $K(q_{s^\frown 0})$ and $K(q_{s^\frown 1})$ are two end-extensions of $K(q_s)$ and $c_{min}(K(q_{s^\frown 0}), K(q_{s^\frown 1})) = i$.
7. Let $n \in \omega$ and assume $\{q_t \mid t \in Y_n\}$, $\{k_l \mid l \leq n\}$, and $\{\dot{p}_t \mid t \in Y_n\}$ have already been constructed. We will now build the items for the nodes in Y_{n+1}. Let $s \in Y_n$, we will define $q_{s^\frown l}, k_{s^\frown l}$ and $\dot{p}_{s^\frown l}$ for $l \in suc_Y(s)$. We are assuming that $\{q_t \mid t \in Y_n\}$ is a (finite) β-fusion tree and $(q_s, \dot{p}_s)$ is (F_n, η_n, k_n, i)-faithful.
   (a) **Case $n \in A_0$** (recall that $F_{n+1} = F_n$, $\eta_{n+1} = \eta_n$ and $suc_Y(s) = \{0\}$).
       Let $\dot{p}_{s^\frown 0} = \dot{p}_s$ and $k_{n+1} = k_n + 3$. We choose $q_{s^\frown 0} \leq^*_{F_n, \eta_n} q_s$ such that $(q_{s^\frown 0}, \dot{p}_{s^\frown 0})$ is $(F_n, \eta_n, k_n + 3, i)$-faithful. This is possible by Lemmas 65 and 74.
   (b) **Case $n \in A_1$** (recall that $F_{n+1} = F_n$, η_{n+1} increases by one in just one point and $suc_Y(s) = \{0\}$).
       Let $q_{s^\frown 0} = q_s$, $k_{n+1} = k_n - 1$ and we find $\dot{p}_{s^\frown 0}$ such that $(q_s, \dot{p}_{s^\frown 0})$ is $(F_n, \eta_{n+1}, k_n - 1, i)$-faithful and $q_s \Vdash "\dot{p}_{s^\frown 0} \leq^{**}_{k_{n+1}} \dot{p}_s."$ This is possible by Lemma 76.

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[6] In this proof, we will identify the nodes in a fusion tree with the value of its last component. As we mentioned earlier, this is possible by Lemma 49.

   (c) **Case $n \in A_2$ and $F_{n+1} = F_n$** (recall that $\eta_{n+1} \upharpoonright F_n = \eta_n$ and $suc_Y(s) = \{0\}$).
Let $q_{s^\frown 0} = q_s$, $k_{n+1} = k_n$ and $\dot{p}_{s^\frown 0} = \dot{p}_s$.

   (d) **Case $n \in A_2$ and $F_{n+1} \neq F_n$** (recall that $|F_{n+1}| = |F_n| + 1$, η_{n+1} takes value 0 in the new point and $suc_Y(s) = \{0, 1\}$).
Let $k_{n+1} = k_n - 1$. We use Lemma 73 to find $\dot{p}_s^0, \dot{p}_s^1$ such that the following conditions hold:

     i. $(q_s, \dot{p}_s^0)$ and $(q_s, \dot{p}_s^1)$ are $(F_n, \eta_n, k_n - 1, i)$-faithful.

     ii. $q_s \Vdash$ "$\dot{p}_s^0, \dot{p}_s^1 \leq^{**}_{k_{n+1}} \dot{p}$."

     iii. $c_{\min}\left(T_n^{\dot{p}_s^0}(q_s * \sigma), T_n^{\dot{p}_s^1}(q_s * \sigma)\right) = i$ whenever $\sigma \in \prod\limits_{\gamma \in F_n} 2^{\eta_n(\gamma)}$.

We now apply Lemma 65 to find $\{q_{t^\frown l} \mid t \in Y_n \wedge l \in 2\}$ a β-fusion tree extending our current tree (in the statement of Lemma 65 we can take $D_t = \mathbb{P}_\beta$ for every $t \in Y_n$). Let $\dot{p}_{s^\frown 0} = \dot{p}_s^0$ and $\dot{p}_{s^\frown 1} = \dot{p}_s^1$.

Now that we have constructed T, it follows by faithfulness that $K(q_s)$ is an i-monochromatic tree for every $s \in Y$. Furthermore, we know that $K(q_{s^\frown 0})$ and $K(q_{s^\frown 1})$ are two end-extensions of $K(p_s)$ and $c_{\min}(K(q_{s^\frown 0}), K(q_{s^\frown 1})) = i$. In this way, $K = \bigcup\limits_{s \in Y} K(q_s)$ is an i-monochromatic tree, which is an element of V.

By Lemma 62, there is $\overline{q} \in \mathbb{P}_\beta$ compatible with T and $\dot{R}$ a name for a branch through T such that $\overline{q}$ forces $\dot{R}$ to be contained in the generic filter. In this way $\overline{p} = Lim(\{\dot{p}_{R\upharpoonright n} \mid n \in \omega\})$ is a name for a condition of $\mathbb{S}(\mathcal{U}_\beta)$ (since it is forced to be the limit of a $\leq^{**}$-decreasing sequence). Finally, $r = \overline{q}^\frown \overline{p}$ is an extension of $q^\frown \dot{p}$ and $r \Vdash$ "$\dot{x} \in [K]$," which is what we desired the most. This finishes the construction and the proof of the successor case. ∎

6 Preserving $c_{\min}$-covering at limit steps

The last task is to prove that the $c_{\min}$-covering is preserved at limit steps of the iteration. For this section, let α be a limit ordinal and $\dot{x}$ a $\mathbb{P}_\alpha$-name for an element of 2^ω that was not added by any $\mathbb{P}_\beta$ for $\beta < \alpha$ (this entails that α has countable cofinality).

Given $i < 2$, define W_i as the set of all $p \in \mathbb{P}_\alpha$ such that for every $\beta < \alpha$ and for every $q \leq p$ there are $q' \leq q$ and q_0, q_1 with the following properties:

1. $q_0 \upharpoonright \beta = q_1 \upharpoonright \beta = q' \upharpoonright \beta$.
2. $q_0, q_1 \leq q'$.
3. $c_{\min}(\dot{x}[q_0], \dot{x}[q_1]) = i$.

The following is Lemma 30 of [9] (in [9], the lemma is only stated for the iteration of Sacks forcing, but it is mentioned that it is true for any iteration, see also Lemma 6.11 of [7]).

Lemma 78 [9]   *Both W_0 and W_1 are open and $W_0 \cup W_1$ is an open dense set.*

We will need the following concepts:

Definition 79   Let $F \in [\alpha]^{<\omega}$, $\eta : F \longrightarrow \omega$, $i < 2$, $\beta = max(f) + 1$ and $p \in \mathbb{P}_\alpha$.

1. We say p is (F, η, i)-*faithful* if $p \in W_i$ and for every $\sigma, \sigma' \in \prod\limits_{\gamma \in F} 2^{\eta(\gamma)}$ such that $\sigma \neq \sigma'$

then $c_{\min}(\dot{x}[p * \sigma], \dot{x}[p * \sigma']) = i$.

2. We say p is (F, η, i)-*splitting at β* if p is (F, η, i)-faithful and there are p^0, p^1 such that:
   (a) $p^0, p^1 \leq p$.
   (b) $p^0 \upharpoonright \beta = p^1 \upharpoonright \beta = p \upharpoonright \beta$.
   (c) If $\sigma \in \prod\limits_{\gamma \in F} 2^{\eta(\gamma)}$, then $\mathsf{c}_{\min} \left(\dot{x} \left[p^0 * \sigma \right], \dot{x} \left[p^1 * \sigma \right] \right) = i$.

In the above case, we say that p^0 and p^1 witness that p is (F, η, i)-splitting at β. The following are some simple remarks:

Lemma 80   Let $F \in [\alpha]^{<\omega}$, $\eta : F \longrightarrow \omega$, $i < 2$, and $p \in \mathbb{P}_\alpha$.

1. If $p \in W_i$ then p is $(\emptyset, \emptyset, i)$-faithful.
2. If p is (F, η, i)-faithful and $q \leq^*_{F,\eta} p$, then q is (F, η, i)-faithful.
3. If $\xi \notin F$ and p is (F, η, i)-faithful then p is $(F \cup \{\xi\}, \eta \cup \{(\xi, 0)\}, i)$-faithful.

It does not seem that the set of all (F, η, i)-splitting conditions is $(F, \eta)^*$-open (below an (F, η, i)-splitting condition). Nevertheless, they are open when we restrict to an ordinal smaller than α.

Lemma 81   Let $F \in [\alpha]^{<\omega}$ with $\beta = \max(f) + 1$, $\eta : F \longrightarrow \omega$, $i < 2$ and $p \in \mathbb{P}_\alpha$ be (F, η, i)-faithful. The set $D \subseteq \mathbb{P}_\beta$ of all $r \in \mathbb{P}_\beta$ for which there is $\overline{r} \in \mathbb{P}_\alpha$ with the following properties:

1. $\overline{r} \upharpoonright \beta = r$.
2. $\overline{r} \leq^*_{F,\eta} p$ (so $r \leq^*_{F,\eta} p \upharpoonright \beta$).
3. $\overline{r}$ is (F, η, i)-splitting at β.

Is an $(F, \eta)^*$-open dense below $p \upharpoonright \beta$.

Proof   We first prove that D is $(F, \eta)^*$-open below $p \upharpoonright \beta$. Let $r \in D$ and $q \in \mathbb{P}_\beta$ such that $q \leq^*_{F,\eta} r$. Let $\overline{r}$ as above and $\overline{r}_0, \overline{r}_1$ witness that $\overline{r}$ is (F, η, i)-splitting at β. Define

1. $\overline{q} = q \char`^ \overline{r} \upharpoonright [\beta, \alpha)$.
2. $\overline{q}_0 = q \char`^ \overline{r}_0 \upharpoonright [\beta, \alpha)$.
3. $\overline{q}_1 = q \char`^ \overline{r}_1 \upharpoonright [\beta, \alpha)$.

Note that $\overline{q} \leq^*_{F,\eta} \overline{r}$ as well as $\overline{q}_0 \leq^*_{F,\eta} \overline{r}_0$ and $\overline{q}_1 \leq^*_{F,\eta} \overline{r}_1$. Since $\overline{q} \leq^*_{F,\eta} \overline{r}$ it follows that $\overline{q}$ is (F, η, i)-faithful. It is easy to see that $\overline{q}_0, \overline{q}_1$ witness that $\overline{q}$ is (F, η, i)-splitting at β.

We will now prove that D is $(F, \eta)^*$-dense below $p \upharpoonright \beta$. Take an enumeration $\prod\limits_{\gamma \in F} 2^{\eta(\gamma)} = \{\sigma_j \mid j \leq k\}$. We will recursively construct $r_j, \overline{q}_j, q^0_j, q^1_j$ for $j \leq k$ such that the following conditions hold:

1. $p = r_0$.
2. $r_{j+1} \leq_{F,\eta} r_j$ for $j + 1 \leq k$.
3. $\overline{q}_j \leq r_j * \sigma_j$.
4. $q^0_j, q^1_j \leq \overline{q}_j$.
5. $q^0_j \upharpoonright \beta = q^1_j \upharpoonright \beta = \overline{q}_j \upharpoonright \beta$.
6. $\mathsf{c}_{\min}(\dot{x}[q^0_j], \dot{x}[q^1_j]) = i$.
7. $r_{j+1} * \sigma_j = \overline{q}_j$ for $j + 1 \leq j$.

The construction is as follows: we start with $p = r_0$. Since $r_0 * \sigma_0 \leq p$ and $p \in W_i$, there is $\overline{q}_0 \leq r_0 * \sigma_0$ and $q_0^0, q_0^1 \leq \overline{q}_0$ such that $q_0^0 \upharpoonright \beta = q_0^1 \upharpoonright \beta = \overline{q}_0 \upharpoonright \beta$ and $c_{\min}(\dot{x}[q_0^0], \dot{x}[q_0^1]) = i$. Now, assume we are at step j, we will define the items at step $j + 1$. We first find r_{j+1} such that $r_{j+1} \leq_{F,\eta} r_j$ and $r_{j+1} * \sigma_j = \overline{q}_j$. Since $r_{j+1} * \sigma_{j+1} \leq r_{j+1}$ and $r_{j+1} \in W_i$, we know that there is $\overline{q}_{j+1} \leq r_{j+1} * \sigma_{j+1}$ and $q_{j+1}^0, q_{j+1}^1 \leq \overline{q}_{j+1}$ such that $q_{j+1}^0 \upharpoonright \beta = q_{j+1}^1 \upharpoonright \beta = \overline{q}_{j+1} \upharpoonright \beta$ and $c_{\min}(\dot{x}[q_{j+1}^0], \dot{x}[q_{j+1}^1]) = i$.

Now, let q such that $q \leq_{F,\eta} r_k$ and $q * \sigma_k = \overline{q}_k$. We claim that $q * \sigma_j \leq \overline{q}_j$ for every $j \leq k$. The statement is clear for $j = k$. Given $j < k$, we have that $q \leq_{F,\eta} r_{j+1}$, so $q * \sigma_j \leq r_{j+1} * \sigma_j = \overline{q}_j$ and we are done.

By the definition by cases lemma (Lemma 45), we can find $\dot{a}, \dot{b}$ two $\mathbb{P}_\beta$-names such that if $j \leq k$, then the following holds:

1. $q * \sigma_j \Vdash ``\dot{a} = q_j^0 \upharpoonright [\delta, \alpha)$."
2. $q * \sigma_j \Vdash ``\dot{b} = q_j^1 \upharpoonright [\delta, \alpha)$."

We now define $\overline{p} = (q \upharpoonright \beta)^\frown p \upharpoonright [\beta, \alpha)$. We claim that $\overline{p}$ is (F, η, i)-splitting at β. Let $p_0 = (q \upharpoonright \beta)^\frown \dot{a}$ and $p_1 = (q \upharpoonright \beta)^\frown \dot{b}$, we will show that p_0 and p_1 have the desired properties.

It is clear that $p_0, p_1 \leq \overline{p}$, and $p_0 \upharpoonright \beta = p_1 \upharpoonright \beta = \overline{p} \upharpoonright \beta$. Now, let $\sigma \in \prod_{\gamma \in F} 2^{\eta(\gamma)}$, we need to prove that $c_{\min}(\dot{x}[p_0 * \sigma], \dot{x}[p_1 * \sigma]) = i$. Let $j \leq k$ such that $\sigma = \sigma_j$. Since $q * \sigma_j \leq \overline{q}_j$, we get that $(p_0 * \sigma_j) \upharpoonright \beta \leq \overline{q}_j \upharpoonright \beta = q_j^0 \upharpoonright \beta$ and $(p_1 * \sigma_j) \upharpoonright \beta \leq \overline{q}_j \upharpoonright \beta = q_j^1 \upharpoonright \beta$. It follows by the definition of $\dot{a}$ and $\dot{b}$ that $p_0 * \sigma_j \leq q_j^0$ and $p_1 * \sigma_j \leq q_j^1$. Since $c_{\min}(\dot{x}[q_j^0], \dot{x}[q_j^1]) = i$, we get that $c_{\min}(\dot{x}[p_0 * \sigma], \dot{x}[p_1 * \sigma]) = i$. ∎

We will now prove the following:

Lemma 82 *Let $F \in [\alpha]^{<\omega}$ with $\beta = \max(f) + 1, \eta : F \longrightarrow \omega$, $i < 2$ and $p \in \mathbb{P}_\alpha$ be (F, η, i)-splitting at β. Pick $\xi \in F$ and define $\eta' : F \longrightarrow \omega$ where $\eta'(\xi) = \eta(\xi) + 1$ and $\eta'(\gamma) = \eta(\gamma)$ whenever $\gamma \neq \xi$. There is $r \in \mathbb{P}_\alpha$ with the following properties:*

1. *$r \upharpoonright \beta = p \upharpoonright \beta$.*
2. *r is (F, η', i)-faithful.*

Proof We know that p is (F, η, i)-splitting at β. Let p^0 and p^1 witness that p is (F, η, i)-splitting at β. For every $\sigma \in \prod_{\gamma \in F} 2^{\eta(\gamma)}$ and for each $j < 2$ let $\sigma_j \in \prod_{\gamma \in F} 2^{\eta'(\gamma)}$ such that $\sigma_j(\beta) = \sigma(\beta)^\frown j$ and $\sigma_j(\gamma) = \sigma(\gamma)$ for every $\gamma \neq \beta$. Let $\dot{a}$ be a $\mathbb{P}_\beta$-name such that $(p * \sigma_j) \upharpoonright \beta \Vdash ``\dot{a} = p^j \upharpoonright [\beta, \alpha)$" for every $\sigma \in \prod_{\gamma \in F} 2^{\eta(\gamma)}$ and for each $j < 2$ (such name exists by the definition by cases lemma (Lemma 45)). The condition $r = (p \upharpoonright \beta)^\frown \dot{a}$ has the desired properties. ∎

We can now prove that the $c_{\min}$-covering is preserved at limit steps:

Proposition 83 *Let α be a limit ordinal and assume $\dot{x}$ is a $\mathbb{P}_\alpha$-name for a real that was not added by $\mathbb{P}_\beta$ for every $\beta < \alpha$. If $p \in \mathbb{P}_\alpha$, then there are $q \leq p$ and $S \in V$ a $c_{\min}$-monochromatic tree such that $q \Vdash ``\dot{x} \in [S]$."*

Proof By Lemma 78, we may assume that there is $i < 2$ such that $p \in W_i$. Let M be a countable elementary submodel of $H(\kappa)$ (for a big large enough κ) such that

$\alpha, \mathbb{P}_\alpha, p, \dot{x} \in M$. Let $L = \{(F_n, \eta_n) \mid n \in \omega\}$ be suitable for (M, α) with the following properties:

For every $l < 4$, define $A_l = \{n \mid n =_4 l\}$ (where $n =_4 l$ means that n is equal to l mod 4). We now find $L = \{(F_n, \eta_n) \mid n \in \omega\}$ suitable for (M, α) with the following extra properties:

1. $F_0 = \emptyset$ and $\eta_0 = \emptyset$.
2. Let $n \in \omega$ and assume F_n and η_n are already defined. We wish to define F_{n+1} and η_{n+1}.
   (a) **Case** $n \in A_0$.
       Let $F_{n+1} = F_n$ and $\eta_{n+1} = \eta_n$.
   (b) **Case** $n \in A_1$.
       Let $F_{n+1} = F_n$ and there is a unique $\xi \in F_n$ such that $\eta_{n+1}(\xi) = \eta_n(\xi) + 1$ while $\eta_{n+1}(\gamma) = \eta_n(\gamma)$ in case $\gamma \neq \xi$.
   (c) **Case** $n \in A_2$.
       Let $F_{n+1} = F_n$ and $\eta_{n+1} = \eta_n$.
   (d) **Case** $n \in A_3$.
       In here, $|F_{n+1}| = |F_n| + 1$, $\eta_{n+1} \restriction F_n = \eta_n \restriction F_n$ and in the new point η_{n+1} takes value 0.

We now recursively define a tree $Y \subseteq 2^{<\omega}$ as follows:

1. $\emptyset \in Y$.
2. Let $s \in Y_n$.
   (a) If $F_{n+1} = F_n$, then $s{}^\frown 0$ is the only immediate successor of s in Y.
   (b) If $F_{n+1} \neq F_n$, then $s{}^\frown 0, s{}^\frown 1 \in Y$.

We now recursively construct an α-fusion tree $T = \{p_s \mid s \in Y\} \subseteq \mathbb{P}_\alpha \cap M$ with the following properties[7]:

1. $p_\emptyset \leq p$.
2. If $s \in Y_n$, then p_s is (F_n, η_n, i)-faithful. Define $K(p_s) = \bigcup\{\dot{x}[p_s * \sigma] \mid \sigma \in \prod_{\gamma \in F_n} 2^{\eta_n(\gamma)}\}$.
3. If $s \in Y_n$ and $j \in suc_Y(s)$, then $p_{s{}^\frown j} \leq^*_{F_n, \eta_n} p_s$.
4. If $s \in Y$ and $j \in suc_Y(s)$, then $K(p_{s{}^\frown j})$ is an end-extension of $K(p_s)$.
5. If $s \in Y$ and $s{}^\frown 0, s{}^\frown 1 \in Y$, then $K(p_{s{}^\frown 0})$ and $K(p_{s{}^\frown 1})$ are two end-extensions of $K(p_s)$ and $c_{\min}(K(p_{s{}^\frown 0}), K(p_{s{}^\frown 1})) = i$.
6. Let $n \in \omega$ and assume $\{p_t \mid t \in Y_n\}$ has already been constructed. We will now build the items for the nodes in Y_{n+1}. Let $s \in Y_n$, we will define $p_{s{}^\frown l}$ for $l \in suc_Y(s)$. We are are assuming that $\{p_t \mid t \in Y_n\}$ is a (finite) α-fusion tree and p_s is (F_n, η_n, i)-faithful. Let $\beta = \max(F_n) + 1$.
   (a) **Case** $n \in A_0$ (recall that $F_{n+1} = F_n$, $\eta_{n+1} = \eta_n$ and $suc_Y(s) = \{0\}$).
       We want $p_{s{}^\frown 0}$ to be (F_{n+1}, η_{n+1}, i)-splitting at β. This is possible by Lemmas 65 and 81.
   (b) **Case** $n \in A_1$ (recall that $F_{n+1} = F_n$, η_{n+1} increases by one in just one point and $suc_Y(s) = \{0\}$).

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[7] Once again, we will identify the nodes in a fusion tree with the value of its last component. (see Lemma 49).

In here, we know that $n - 1 \in A_0$, so by the previous case we know that p_s is (F_n, η_n, i)-splitting at β. We want $p_{s^\frown 0}$ such that $p_{s^\frown 0} \restriction \beta = p_s \restriction \beta$ and $p_{s^\frown 0}$ is (F_{n+1}, η_{n+1}, i)-faithful. This is possible by Lemma 82.

(c) **Case $n \in A_2$** (recall that $F_{n+1} = F_n$, $\eta_{n+1} = \eta_n$ and $suc_Y(s) = \{0\}$).
   We want $p_{s^\frown 0}$ to be (F_{n+1}, η_{n+1}, i)-splitting at β. This is possible by and Lemmas 65 and 81 (this point is the same as in the case where $n \in A_0$).

(d) **Case $n \in A_3$** (recall that $|F_{n+1}| = |F_n| + 1$, $\eta_{n+1} \restriction F_n = \eta_n \restriction F_n$ in the new point η_{n+1} takes value 0 and $suc_Y(s) = \{0, 1\}$).
   In here, we know that $n - 1 \in A_2$, so by the previous case, we know that p_s is (F_n, η_n, i)-splitting at β. For each $s \in Y_n$, choose p_s^0 and p_s^1 that witness that p_s is (F_n, η_n, i)-splitting at β.
   Let $\widetilde{T} = \{p_t \restriction \beta \mid t \in Y_{\leq n}\}$, which we know is a β-fusion tree. Apply Lemma 65 to find $\{\overline{p}_{t^\frown l} \mid t \in Y_n \wedge l \in 2\}$ a β-fusion tree extending $\widetilde{T}$ (in the statement of Lemma 65, we can take $D_t = \mathbb{P}_\beta$ for every $t \in Y_n$). Define $p_{s^\frown 0} = \widehat{\overline{p}_{s^\frown 0} p_s^0} \restriction [\beta, \alpha)$ and $p_{s^\frown 1} = \widehat{\overline{p}_{s^\frown 1} p_s^1} \restriction [\beta, \alpha)$.

Using the faithfulness of the conditions, it is easy to see that $K(p_s)$ is an i-monochromatic tree. Furthermore, we know that $K(p_{s^\frown 0})$ and $K(p_{s^\frown 1})$ are two end-extensions of $K(p_s)$ and $c_{\min}(K(p_{s^\frown 0}), K(p_{s^\frown 1})) = i$. In this way, $K = \bigcup_{s \in Y} K(p_s)$ is an i-monochromatic tree, which is an element of V. By Lemma 62, there is $\overline{p} \in \mathbb{P}_\alpha$ compatible with T and $\dot{R}$ a name for a branch through T such that $\overline{p}$ forces $\dot{R}$ to be contained in the generic filter. It follows that $\overline{p} \Vdash$ "$\dot{x} \in [K]$," which is what we wanted. ∎

After all our hard work, we can finally prove the main result of the paper:

Theorem 84 *The inequality* ℏm < 𝔲 *is consistent with the axioms of* ZFC.

Proof We start with a model V of the Generalized Continuum Hypothesis. We perform a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \omega_2 \rangle$ such that if $\alpha < \omega_2$ then $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{S}(\dot{\mathcal{U}}_\alpha)$" where $\dot{\mathcal{U}}_\alpha$ is a $\mathbb{P}_\alpha$-name for an ultrafilter. Furthermore, with a carefully chosen book-keeping devise we can make sure that $\mathbb{P}_{\omega_2} \Vdash$ "$\mathfrak{u} = \omega_2$." By the results in the last sections, we know that $\mathbb{P}_{\omega_2}$ preserves $c_{\min}$-covering, so $\mathbb{P}_{\omega_2} \Vdash$ "ℏm $= \omega_1$." ∎

7 MAD **families in the Shelah model**

By the Shelah model, we mean an iteration as in the last theorem. We already know that ℏm $= \omega_1$ holds in such model, so all the cardinal invariants in the Cichoń diagram are small. Furthermore, in that model $\mathfrak{u} = \omega_2$ and $\mathfrak{i} = \omega_1$ (see [13]). The only one of the usual cardinal invariants that is missing to compute $\mathfrak{a}$. In this last section, we will prove that there is a MAD family of size ω_1 in the Shelah model. As mentioned in the introduction, in recent work with Cruz-Chapital et al., we obtained a different proof of this result (see [12]).

Let $f, g \in \omega^\omega$, define $f \leq^* g$ if and only if $f(n) \leq g(n)$ for all $n \in \omega$ except finitely many. A family $\mathcal{D} \subseteq \omega^\omega$ is a *dominating family* if for every $f \in \omega^\omega$, there is $g \in \mathcal{D}$ such that $f \leq^* g$. The *dominating number* $\mathfrak{d}$ is defined as the least size of a dominating family. In [10], Hrušák introduced a diamond principle for the dominating number:

$\diamondsuit_{\mathfrak{d}}$ There is a sequence $\langle d_\alpha \mid \alpha < \omega_1 \rangle$ where $d_\alpha : \alpha \longrightarrow \omega$ such that for every $f : \omega_1 \longrightarrow \omega$ the set $\{\alpha > \omega \mid f \restriction \alpha \leq^* d_\alpha\}$ is nonempty. The sequence is called a $\diamondsuit_{\mathfrak{d}}$-*sequence.*

Above, $f \restriction \alpha \leq^* d_\alpha$ means that the set $\{\xi < \alpha \mid d_\alpha(\xi) < f(\xi)\}$ is finite. It is easy to see that $\diamondsuit_\delta$ implies that $\mathfrak{d} = \omega_1$. It is an old problem of Roitman if $\mathfrak{d} = \omega_1$ implies $\mathfrak{a} = \omega_1$, however, Hrušák proved the following:

Proposition 85 [10] $\diamondsuit_{\mathfrak{d}}$ *implies* $\mathfrak{a} = \omega_1$.

In order to show that $\mathfrak{a} = \omega_1$ holds in the Shelah model, we will prove that $\diamondsuit_{\mathfrak{d}}$ is true in there. Recall that $\diamondsuit$ is the following statement:

$\diamondsuit$ There is $\mathcal{D} = \{D_\alpha \mid \alpha \in \omega_1\}$ with $D_\alpha \subseteq \alpha$ such that for every $X \subseteq \omega_1$, the set $\{\alpha \mid X \cap \alpha = D_\alpha\}$ is stationary.

We will use $\diamondsuit$ in order to construct a $\diamondsuit_{\mathfrak{d}}$-sequence for the Shelah model. By $LIM(\omega_1)$, we denote the set of all countable limit ordinals. We start with the following:

Lemma 86   *Let* $V \vDash \diamondsuit$ *and* κ *a large enough regular cardinal. There is a sequence* $\langle (M_\alpha, p_\alpha, \dot{f}_\alpha) \rangle_{\alpha \in LIM(\omega_1)}$ *such that for every* $\alpha < \omega_1$ *the following holds:*

1. M_α *is a countable elementary submodel of* $H(\kappa)$ *such that* $\mathbb{P}_{\omega_2}, p_\alpha, \dot{f}_\alpha \in M_\alpha$ *(where* $\mathbb{P}_{\omega_2}$ *is the iteration of the Shelah forcing).*
2. $p_\alpha \in \mathbb{P}_{\omega_2}$ *and* $p_\alpha \Vdash$ "$\dot{f}_\alpha : \omega_1 \longrightarrow \omega$."

With the property that for every $p \in \mathbb{P}_{\omega_2}$ *and* $\dot{f}$ *such that* $p \Vdash$ "$\dot{f} : \omega_1 \longrightarrow \omega$," *there are countable* $N \preceq H(\kappa)$ *and* $\alpha < \omega_1$ *such that the following conditions hold:*

1. $\mathbb{P}_{\omega_2}, p, \dot{f} \in N$.
2. $M_\alpha \cap \omega_1 = \alpha$.
3. *The structures* $(N, \in, \mathbb{P}_{\omega_2}, p, \dot{f})$ *and* $(M_\alpha, \in, \mathbb{P}_{\omega_2}, p_\alpha, \dot{f}_\alpha)$ *are isomorphic.*

Proof   Using $\diamondsuit$, we can find a sequence $\langle \mathfrak{A}_\alpha = (\alpha, \rhd_\alpha, P_\alpha, r_\alpha, h_\alpha) \rangle_{\alpha \in LIM(\omega_1)}$ such that for every structure $\mathfrak{A} = (\omega_1, \rhd, P, r, h)$ there are stationary many α such that $\mathfrak{A}_\alpha$ is a substructure of $\mathfrak{A}$. Given α a limit ordinal, in case there are a countable $M \preceq H(\kappa)$, $p \in \mathbb{P}_{\omega_2}, \dot{f}$ such that $\mathbb{P}_{\omega_2}, p, \dot{f} \in M$, $M \cap \alpha = \alpha$, $p \Vdash$ "$\dot{f} : \omega_1 \longrightarrow \omega$" and $(M, \in, \mathbb{P}_{\omega_2}, p, \dot{f})$ is isomorphic to $\mathfrak{A}_\alpha$ then we choose one of them and define $M_\alpha = M$, $p_\alpha = p$ and $\dot{f}_\alpha = \dot{f}$. If there is no M satisfying those properties, we just take any $(M_\alpha, p_\alpha, \dot{f}_\alpha)$ satisfying the properties 1 and 2. We will now prove $\mathcal{D} = \{(M_\alpha, p_\alpha, \dot{f}_\alpha) \mid \alpha \in LIM(\omega_1)\}$ has the desired properties.

Let $p \in \mathbb{P}_{\omega_2}$ and $\dot{f}$ such that $p \Vdash$ "$\dot{f} : \omega_1 \longrightarrow \omega$." Recursively, we build $\{N_\alpha \mid \alpha < \omega_1\}$ a continuous $\in$-chain of countable elementary submodels of $H(\kappa)$ such that $p, \dot{f}, \mathbb{P}_{\omega_2} \in N_0$. Let $N = \bigcup_{\alpha \in \omega_1} N_\alpha$, since N has size ω_1 then we can define a structure $\mathfrak{A} = (\omega_1, \rhd, P, r, h)$ that is isomorphic to $(N, \in, \mathbb{P}_{\omega_2}, p, \dot{f})$. Let $F : \omega_1 \longrightarrow N$ be an isomorphism.

It is easy to see that $\{\alpha \in LIM(\omega_1) \mid N_\alpha \cap \omega_1 = \alpha \wedge F[\alpha] = N_\alpha\}$ is a club. In this way, we can find α such that $F[\alpha] = N_\alpha$, $N_\alpha \cap \omega_1 = \alpha$ and $\mathfrak{A}_\alpha$ is a substructure of $\mathfrak{A}$. Note that N_α, p and $\dot{f}$ satisfy the conditions of the definition at step α, so $(M_\alpha, \in, \mathbb{P}_{\omega_2}, p_\alpha, \dot{f}_\alpha)$ is isomorphic to $\mathfrak{A}_\alpha$ hence it is also isomorphic to $(N, \in, \mathbb{P}_{\omega_2}, p, \dot{f})$ (of course it might be the case $M_\alpha = N_\alpha$ but this is highly unlikely).∎

With the lemma, we can now prove the following:

Proposition 87 $\diamondsuit_\partial$ *holds in the Shelah model.*

Proof   It is well known that we may assume that $\diamondsuit$ holds in V. Fix a sequence $\langle (M_\alpha, p_\alpha, \dot{f}_\alpha) \rangle_{\alpha \in LIM(\omega_1)}$ as in the previous lemma, we will now define $\mathcal{D} = \{ d_\alpha : \alpha \longrightarrow \omega \mid \alpha < \omega_1 \}$. In case $M_\alpha \cap \omega_1 \neq \alpha$, let d_α be any constant function. Fix α such that $M_\alpha \cap \omega_1 = \alpha$, we will see how to define d_α.

Let $L = \{ (F_n, \eta_n) \mid n \in \omega \}$ that is (M_α, ω_2)-suitable. Fix an enumeration $\alpha = \{ \alpha_n \mid n \in \omega \}$. For every $\beta < \alpha$ define $D_\beta^\alpha = \{ q \mid \exists n (q \Vdash \text{``} \dot{f}_\alpha(\beta) = n\text{''}) \}$ and recall that the set $\widetilde{D_\beta^\alpha}(F, \eta) = \{ q \mid \forall \sigma \in \prod_{\delta \in F} 2^{\eta(\delta)} \exists n_\sigma (q * \sigma \Vdash \text{``} \dot{f}_\alpha(\beta) = n_\sigma''\text{''}) \}$ is $(F, \eta)^*$-open dense and it is an element of M_α. We can build a fusion tree $T = \{ p_s \mid s \in 2^{<\omega} \}$ with the following properties:

1. $p_\emptyset = p_\alpha$.
2. If $s \in 2^{n+1}$ then $p_s \in \widetilde{D_{\beta_n}^\alpha}(F_n, \eta_n)$.

Define $d_\alpha : \alpha \longrightarrow \omega$ such that if $n \in \omega$ then $p_s \Vdash \text{``} \dot{f}_\alpha(\alpha_n) < d_\alpha(\alpha_n)\text{''}$ for every $s \in 2^{n+1}$. We will prove that $\mathcal{D} = \{ d_\alpha : \alpha \longrightarrow \omega \mid \alpha < \omega_1 \}$ will be a $\diamondsuit_\partial$-sequence after forcing with $\mathbb{P}_{\omega_2}$. Let $p \in \mathbb{P}_{\omega_2}$ and $\dot{f}$ such that $p \Vdash \text{``} \dot{f} : \omega_1 \longrightarrow \omega.\text{''}$ Applying the previous lemma, we can find a countable N for which $\mathbb{P}_{\omega_2}, p, \dot{f} \in N$ and $\alpha < \omega_1$ such that $M_\alpha \cap \omega_1 = \alpha$ and the structures $(N, \in, \mathbb{P}_{\omega_2}, p, \dot{f})$ and $(M_\alpha, \in, \mathbb{P}_{\omega_2}, p_\alpha, \dot{f}_\alpha)$ are isomorphic. Let $H : M_\alpha \longrightarrow N$ be the isomorphism. Let $F_n' = H(F_n)$ and $\eta_n' = H(\eta_n)$. Let $L' = \{ (F_n', \eta_n') \mid n \in \omega \}$. For every $s \in 2^{<\omega}$ define $p_s' = H(p_s)$. In this way, $p_\emptyset' = p$ and it is easy to see that $T' = \{ p_s' \mid s \in 2^{<\omega} \}$ is a fusion tree.

Let q be compatible with T' (which is obviously an extension of p). We will now prove that $q \Vdash \text{``} \dot{f} \restriction \alpha \leq d_\alpha.\text{''}$ Let $n \in \omega$ and $s \in 2^{n+1}$, since $p_s \Vdash \text{``} \dot{f}_\alpha(\alpha_n) < d_\alpha(\alpha_n)\text{''}$ it follows that $p_s' \Vdash \text{``} \dot{f}(\alpha_n) < d_\alpha(\alpha_n).\text{''}$ Finally, since $q$ is compatible with $T'$, it follows that $\{ p_s' \mid s \in 2^{n+1} \}$ is predense below $q$, so $q \Vdash \text{``} \dot{f}(\beta_n) < d_\alpha(\beta_n).\text{''}$ ∎

8   Open questions

We do not know the answer to the following:

Problem 88   Can a generic filter for $\mathbb{S}(\mathcal{U})$ be reconstructed from the generic real?

Much of the work in this paper would be simplified if $\mathbb{S}(\mathcal{U})$ was an Axiom A forcing, so we could ask the following:

Problem 89   Let $\mathcal{U}$ be an ultrafilter. Is it possible to give $\mathbb{S}(\mathcal{U})$ an Axiom A structure?

Unfortunately, we conjecture that this is not possible. An iteration theorem would greatly simplify the work in this paper, as well as in [7, 9]. We could ask the following:

Problem 90   If $\mathbb{P}$ preserves c_{min}-covering and $\mathbb{P} \Vdash$“$\dot{\mathbb{Q}}$ preserves c_{min}-covering, ” is it true that $\mathbb{P} * \dot{\mathbb{Q}}$ preserves c_{min}-covering?

Problem 91   If δ is a limit ordinal, $\langle \mathbb{P}_\alpha, \dot{\mathbb{R}}_\alpha \mid \alpha < \delta \rangle$ is a countable support iteration of proper forcings such that each $\mathbb{P}_\alpha$ preserves c_{min}-covering, is it true that $\mathbb{P}_\delta$ preserves c_{min}-covering?

It might be possible to use the results from chapter VI of [14] to prove an iteration theorem for preserving c_{min}-covering, but we were unable to do so.

We do not know the answer to the following question:

Problem 92   Are there *P*-points in the Shelah model of $\mathfrak{i} < \mathfrak{u}$?

We conjecture that the answer is negative. It might be possible that similar ideas to the ones used in [6] can be used to prove that there are no *P*-points in the Shelah model, but we were unable to prove it.

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